PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

# Diffraction

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*Phil. Trans. R. Soc. Lond. A* 1930 **229**, 87-124
doi: 10.1098/rsta.1930.0003

# III. *Diffraction.*

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(*Communicated by* F. E. SMITH, SEC.R.S.)

(Received November 14, 1929—Read January 23, 1930.)

CONTENTS.



## INTRODUCTION.

When HUYGEN'S principle is applied to the problem of the straight edge, FRESNEL'S diffraction phenomena in the neighbourhood of the geometrical shadow can be accounted for, and the theory agrees closely with observation.

But so many approximations are involved in the application of FRESNEL'S theory, that an outstanding event in the history of diffraction theory was the discovery of the exact solution for waves impinging upon a semi-infinite plane.

This problem constitutes the only one in diffraction theory which has been solved completely in a comparatively simple form. It is a special case of the wedge problem, the successful treatment of which is due to the fact that there are no dimensions concerned which bear a relation to the wave-length of the incident disturbance. The solution of the problem is due to the labours of a number of mathematicians, among whom POINCARÉ ('Acta Math.,' vol. 16, p. 297 (1892–3)), SOMMERFELD ("Math. Theorie der Diffraction," 'Math. Ann.,' vol. 47, pp. 317–374 (1895)), MACDONALD ("Electric Waves," and 'Proc. London Math. Soc.,' ser. 2, vol. 14, part 6), and BROMWICH (*ibid.*), may be mentioned.

Now, when the edge of the plane is not infinitely thin, or, more precisely, when it is so fashioned that dimensions are involved which are comparable with the wave-length of the incident disturbance, there must be some additional effects upon the diffraction phenomena in the neighbourhood of the geometrical shadow. The discussion of these effects is one of the subjects of the present paper, for FRESNEL'S theory cannot be

VOL. CCXXIX.—A 672 N [Published, April 17, 1930.

considered as complete without some further investigation. It is shown that the problem of the diffraction of waves by two parallel semi-infinite planes can be solved, one of the most interesting applications of the solution being made to a consideration of the effect of the thickness of a straight edge upon FRESNEL's diffraction phenomena. Further, the problem of a train of waves impinging upon a thin two-dimensional lamina is one of such difficulty that it has not been treated with complete success. But if the lamina be joined to a thin semi-infinite plane partition, the two being at right angles to one another and the centre line of the former being coincident with the edge of the latter, the problem becomes tractable. From the acoustical point of view this latter problem is of considerable importance. The introduction of the partition prevents complete circulation around the lamina, and thus simplifies the analytical treatment. The effect of the lamina upon FRESNEL's diffraction phenomena is discussed. Again, let us consider a tube, into which waves are entering. In the ideal case of a tube with infinitely thin perfectly reflecting walls, consisting of two parallel semi-infinite planes and thus constituting the two-dimensional example, the problem can be solved completely. Apart from the applications mentioned, the analytical method is given in some detail, for it is capable of giving, by the repeated application of a simple though probably laborious process, the complete solution of the problems considered.

Although the analysis in these problems is somewhat long and intricate it furnishes some interesting results in the theory of infinite series, and it is hoped that the practical solutions, of which unfortunately there are only too few in this important branch of mathematical physics, may be of some value.

## SECTION 1.—PRELIMINARY THEORY.

The diffracting surfaces considered are, in every case, two-dimensional. In other words their generating lines are parallel to some fixed direction in space, which is assumed to be the direction of the axis of $z$.

When the waves incident upon a surface are plane waves of sound, suitable solutions will be obtained, and in this case a function $u_0$ must be found whose normal gradient vanishes at every point of the surface.

The undisturbed incident wave requires precise definition if it is electromagnetic. BROMWICH has shown, in his paper referred to in the Introduction, that the problem of diffraction can be solved in the case of a perfectly reflecting wedge, if the electromagnetic waves proceed from a Hertzian oscillator whose axis is parallel to the edge of the wedge.

It is not difficult to show that the problem can be solved in the case of any two-dimensional surface with arbitrary optical properties, if the axis of the oscillator is parallel to the generating lines of the surface.

Accordingly the incident electromagnetic waves which we shall have under consideration are assumed to proceed from a Hertzian oscillator whose axis is parallel to the axis of $z$, the oscillator being at such a great distance from that portion of space to which the analysis is applied that in the said space the undisturbed incident wave may be assumed to be plane. Unless otherwise stated the diffracting surface is assumed to be perfectly reflecting, and then the solution required is expressed by a vector $u_0$ which must vanish at every point of the surface.

If $u_0$ be expressed in cylindrical co-ordinates $r$, $\theta$, $z$, the polarisation at any point in space is obtained from MAXWELL'S equations. Let X, Y, Z be the components of electric force respectively along the radius vector $r$, perpendicular to $r$ and $z$, and along $z$; let $a$, $b$, $c$ be the corresponding components of magnetic force; let $c$ be the velocity of the wave and $t$ the time. Then

$$\left.\begin{aligned} \frac{1}{c^2}\frac{\partial \mathrm{X}}{\partial t} &= \frac{\partial^2 u_0}{\partial r\,\partial z}, & a &= \frac{\partial u_0}{r\,\partial\theta} \\ \frac{1}{c^2}\frac{\partial \mathrm{Y}}{\partial t} &= \frac{\partial^2 u_0}{r\,\partial\,\theta\,\partial z}, & b &= -\frac{\partial u_0}{\partial r} \\ \frac{1}{c^2}\frac{\partial \mathrm{Z}}{\partial t} &= -\frac{1}{c^2}\frac{\partial^2 u_0}{\partial t^2} + \frac{\partial^2 u_0}{\partial z^2}, & c &= 0 \end{aligned}\right\} \quad \ldots \ldots \quad (1)$$

The theory to which we proceed is preferably commenced by employing rectangular co-ordinates. Let $u$ be a wave function, so that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \quad \ldots \ldots \quad (2)$$

Write

$$u = e^{i\kappa(lx+my+nz+ct)}\,.\,\mathrm{F}_0,$$

where

$$l^2 + m^2 + n^2 = 1\,;$$

then, if $\mathrm{F}_0$ be a function of $x$ and $y$ only,

$$\frac{\partial^2 \mathrm{F}_0}{\partial x^2} + \frac{\partial^2 \mathrm{F}_0}{\partial y^2} + 2i\kappa l\frac{\partial \mathrm{F}_0}{\partial x} + 2i\kappa m\frac{\partial \mathrm{F}_0}{\partial y} = 0. \quad \ldots \ldots \quad (3)$$

We shall now transform to other variables $\phi$ and $\psi$ connected with $x$ and $y$ by the complex equation

$$x + iy = f(\phi + i\psi),$$

or

$$z' = f(\chi).$$

We have

$$\frac{\partial \mathrm{F}_0}{\partial x} = \frac{\partial \mathrm{F}_0}{\partial \phi}\cdot\frac{\partial \phi}{\partial x} + \frac{\partial \mathrm{F}_0}{\partial \psi}\cdot\frac{\partial \psi}{\partial x},$$

and a similar equation for $\partial F_0/\partial y$. Writing, in accordance with the well-known theory of the complex variable,

$$\frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = -u' + iv',$$

we have

$$-\frac{\partial z'}{\partial\chi} = -\frac{\partial x}{\partial\phi} - i\frac{\partial y}{\partial\phi},$$

$$= \frac{1}{q^2}(u' + iv'),$$

where

$$q^2 = u'^2 + v'^2.$$

Hence

$$\frac{\partial\phi}{\partial x} = -u' = q^2\frac{\partial x}{\partial\phi} = \frac{\partial\psi}{\partial y},$$

$$\frac{\partial\psi}{\partial x} = v' = -q^2\frac{\partial y}{\partial\phi} = -\frac{\partial\phi}{\partial y},$$

and

$$\frac{\partial^2 F_0}{\partial x^2} + \frac{\partial^2 F_0}{\partial y^2} = q^2\left(\frac{\partial^2 F_0}{\partial\phi^2} + \frac{\partial^2 F_0}{\partial\psi^2}\right).$$

Let $\phi_0$ be the angle which the normal to the undisturbed incident wave front makes with the axis of $z$, then we shall put

$$l = \sin\phi_0 \cos\psi_0,$$

$$m = \sin\phi_0 \sin\psi_0,$$

$$n = \cos\phi_0.$$

Equation (3) accordingly transforms into

$$\frac{\partial^2 F_0}{\partial\phi^2} + \frac{\partial^2 F_0}{\partial\psi^2} + 2i\kappa\sin\phi_0\left\{\cos\psi_0\left(\frac{\partial F_0}{\partial\phi}\cdot\frac{\partial x}{\partial\phi} + \frac{\partial F_0}{\partial\psi}\cdot\frac{\partial x}{\partial\psi}\right)\right.$$
$$\left. + \sin\psi_0\left(-\frac{\partial F_0}{\partial\phi}\cdot\frac{\partial x}{\partial\psi} + \frac{\partial F_0}{\partial\psi}\cdot\frac{\partial x}{\partial\phi}\right)\right\} = 0. \quad . \quad . \quad (4)$$

There is another form of the differential equation which will be found useful. Let

$$u = F' e^{i\kappa(nz+ct)},$$

where $F'$ is a function of $x$ and $y$ only. Then

$$\frac{\partial^2 F'}{\partial x^2} + \frac{\partial^2 F'}{\partial y^2} + \kappa^2(1 - n^2) F' = 0,$$

which, in the transformed co-ordinates, is equivalent to the equation

$$\frac{\partial^2 F'}{\partial \phi^2} + \frac{\partial^2 F'}{\partial \psi^2} + \frac{\kappa^2 (1 - n^2)}{q^2} F' = 0.$$

Let now

$$F' = e^{i\kappa\gamma (l\phi + m\psi)} \, . \, F,$$

so that,

$$u = e^{i\kappa (\gamma l\phi + \gamma m\psi + nz + ct)} \, . \, F,$$

then F must satisfy the equation

$$\frac{\partial^2 F}{\partial \phi^2} + \frac{\partial^2 F}{\partial \psi^2} + 2i\kappa\gamma l \frac{\partial F}{\partial \phi} + 2i\kappa\gamma m \frac{\partial F}{\partial \psi} + \kappa^2 \left(\frac{1 - n^2}{q^2} - \gamma^2 l^2 - \gamma^2 m^2\right) F = 0. \quad . \quad . \quad (5)$$

It is, of course, only for certain transformations that the differential equations (4) and (5) are tractable, and to the consideration of some of these we now proceed.

## Section 2.—The Transformation Appropriate to Two Parallel Semi-Infinite Planes.

Waves are incident in the direction $\psi_0$ $\phi_0$ upon the two parallel semi-infinite planes AA′ and BB′ (fig. 1). It should be noticed that $\psi_0$ is not an angle but a parameter, but the curves, $\psi =$ const., are sensibly straight lines when $\phi$ is large, the parameters $\phi$ and $\psi$ being defined as follows. The transformation required for this problem was

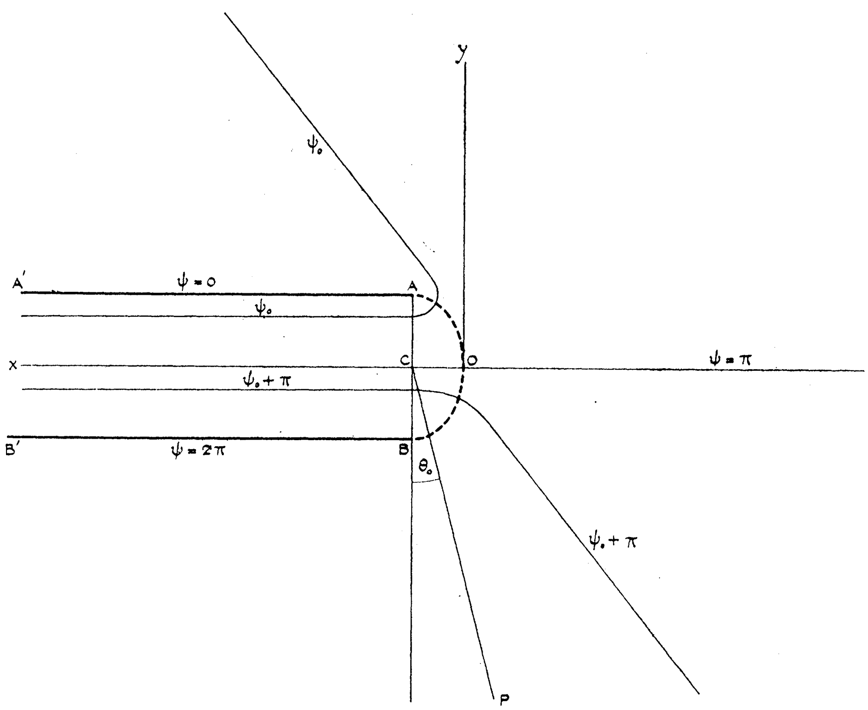


Fig. 1.

first given by HELMHOLTZ, but it is obtainable at once by the application of a general method due to SCHWARZ. The transformation is

$$\left.\begin{aligned} z' &= \frac{a}{\pi}(-\chi + e^{\chi}) + \omega, \\ \text{where} \quad z' &= x + iy, \\ \chi &= \phi + i\psi, \\ \text{and} \quad \omega &= \alpha + i\beta. \end{aligned}\right\} \quad \ldots \ldots \quad (6)$$

In fig. 1 AA′ corresponds to $\psi = 0$ and BB′ corresponds to $\psi = 2\pi$. The broken curve AOB corresponds to $\phi = 0$.

If it be assumed that $x = 0$ and $y = 0$ when $\phi = 0$ and $\psi = \pi$, then it follows that

$$\left.\begin{aligned} x &= \frac{a}{\pi}(1 - \phi + e^{\phi}\cos\psi) \\ \text{and} \quad y &= \frac{a}{\pi}(\pi - \psi + e^{\phi}\sin\psi) \end{aligned}\right\}, \quad \ldots \ldots \quad (7)$$

$2a$ being the distance between the two planes. As in the first section we write

$$u = e^{i\kappa(lx+my+nz+ct)} \cdot F_0,$$

where

$$l = \sin\phi_0 \cos\psi_0,$$

$$m = \sin\phi_0 \sin\psi_0,$$

and

$$n = \cos\phi_0.$$

Now

$$i\kappa(lx + my + nz + ct) = \tfrac{1}{2}A\cos\psi_0(1 - \phi) + \tfrac{1}{2}A\sin\psi_0(\pi - \psi) + \tfrac{1}{2}Ae^{\phi}\cos(\psi - \psi_0) + i\kappa(\cos\phi_0 \cdot z + ct).$$

where

$$A = 2i\kappa\frac{a}{\pi}\sin\phi_0.$$

Hence

$$u = e^{M+N}F_0,$$

where

$$M = \tfrac{1}{2}A\sin\psi_0(\pi - \psi),$$

and

$$N = \tfrac{1}{2}A\cos\psi_0(1 - \phi) + \tfrac{1}{2}Ae^{\phi}\cos(\psi - \psi_0) + i\kappa(\cos\phi_0 \cdot z + ct).$$

The reason for this procedure will be apparent subsequently.

Let now

$$F_0 = e^{-M}F,$$

so that

$$u = e^{N} \cdot F.$$

In order to find the differential equation satisfied by F we have, in the first place,

$$\left.\begin{aligned} \frac{\partial F_0}{\partial\psi} &= e^{-M}\left(\frac{\partial F}{\partial\psi} + \tfrac{1}{2}A\sin\psi_0 \,.\, F\right), \\ \text{and}\quad \frac{\partial^2 F_0}{\partial\psi^2} &= e^{-M}\left(\frac{\partial^2 F}{\partial\psi^2} + A\sin\psi_0\frac{\partial F}{\partial\psi} + \tfrac{1}{4}A^2\sin^2\psi_0 F\right). \end{aligned}\right\} \quad \ldots\ldots \quad (8)$$

In the second place, on applying the differential equation (4) to the transformation (7), it is found that $F_0$ satisfies the equation

$$\frac{\partial^2 F_0}{\partial\phi^2} + \frac{\partial^2 F_0}{\partial\psi^2} - A\frac{\partial F_0}{\partial\phi}\cos\psi_0 - A\frac{\partial F_0}{\partial\psi}\sin\psi_0 + A\frac{\partial F_0}{\partial\phi}e^{\phi}\cos(\psi-\psi_0)$$
$$- A\frac{\partial F_0}{\partial\psi}e^{\phi}\sin(\psi-\psi_0) = 0.$$

On substituting from (8) it follows that F must satisfy the equation

$$\frac{\partial^2 F}{\partial\phi^2} + \frac{\partial^2 F}{\partial\psi^2} - A\frac{\partial F}{\partial\phi}\cos\psi_0 + A\frac{\partial F}{\partial\phi}e^{\phi}\cos(\psi-\psi_0) - A\frac{\partial F}{\partial\psi}e^{\phi}\sin(\psi-\psi_0)$$
$$- \tfrac{1}{4}A^2\sin^2\psi_0 \,.\, F - \tfrac{1}{2}A^2\sin\psi_0 e^{\phi}\sin(\psi-\psi_0) \,.\, F = 0. \quad \ldots\ldots \quad (9)$$

Now the solution of the problem of the single infinitely thin semi-infinite plane, deduced by making $\kappa a$ very small, is obtained from (9) by omitting the third term and the last two terms upon the left-hand side of that equation. It is, in fact, a first approximation to the solution which is being sought. The expansion of F which is, accordingly, suggested is one in ascending powers of $\cos\frac{1}{2}(\psi-\psi_0)$.

Let us write, momentarily,

$$P = \cos^n\tfrac{1}{2}(\psi-\psi_0) = \mu^n, \text{ say};$$

we have then

$$\frac{\partial P}{\partial\psi} = -\tfrac{1}{2}n\mu^{n-1} \,.\, \sin\tfrac{1}{2}(\psi-\psi_0),$$

$$\frac{\partial^2 P}{\partial\psi^2} = \tfrac{1}{4}n(n-1)\mu^{n-2} - \tfrac{1}{4}n^2\mu^n,$$

$$P\cos(\psi-\psi_0) = 2\mu^{n+2} - \mu^n,$$

$$\frac{\partial P}{\partial\psi}\sin(\psi-\psi_0) = n\mu^n + n\mu^{n+2}.$$

Again, writing momentarily,

$$Q = \sin\tfrac{1}{2}(\psi-\psi_0)\,\mu^{n-1},$$

we have

$$\frac{\partial Q}{\partial \psi} = \tfrac{1}{2} n \mu^n - \tfrac{1}{2}(n-1)\mu^{n-2},$$

$$\frac{\partial^2 Q}{\partial \psi^2} = \tfrac{1}{4} \sin \tfrac{1}{2}(\psi - \psi_0)\{(n-1)(n-2)\mu^{n-3} - n^2 \mu^{n-1}\},$$

$$Q \cos(\psi - \psi_0) = \sin \tfrac{1}{2}(\psi - \psi_0)\{2\mu^{n+1} - \mu^{n-1}\},$$

$$\frac{\partial Q}{\partial \psi} \sin(\psi - \psi_0) = \sin \tfrac{1}{2}(\psi - \psi_0)\{n\mu^{n+1} - (n-1)\mu^{n-1}\}.$$

The suitable expansion for F is, therefore,

$$F = \Sigma u_n,$$

where

$$u_n = \mu^n . F_n + \tan \tfrac{1}{2}(\psi - \psi_0) . \mu^n . G_n,$$

$F_n$ and $G_n$ are functions of $\phi$ only, and $n$ may take either all even integral values from 0 to infinity or all odd integral values from 1 to infinity.

If now the left-hand side of (9) be applied to $u_n$, there is obtained the expression

$$\mu^n \left\{\frac{\partial^2 F_n}{\partial \phi^2} - \tfrac{1}{4} n^2 F_n - A \cos \psi_0 \frac{\partial F_n}{\partial \phi} - A e^\phi \frac{\partial F_n}{\partial \phi} + nAe^\phi F_n - \tfrac{1}{4} A^2 \sin^2 \psi_0 F_n - A^2 \sin \psi_0 e^\phi G_n\right\}$$
$$+ \mu^{n-2}\{\tfrac{1}{4} n(n-1) F_n\}$$
$$+ \mu^{n+2}\left\{2Ae^\phi \frac{\partial F_n}{\partial \phi} - nAe^\phi F_n + A^2 \sin \psi_0 e^\phi G_n\right\}$$
$$+ \tan \tfrac{1}{2}(\psi - \psi_0)$$
$$\times \left[\mu^n\left\{\frac{\partial^2 G_n}{\partial \phi^2} - \tfrac{1}{4} n^2 G_n - A \cos \psi_0 \frac{\partial G_n}{\partial \phi} - Ae^\phi \frac{\partial G_n}{\partial \phi} + (n-1)Ae^\phi G_n - \tfrac{1}{4} A^2 \sin^2 \psi_0 G_n\right\}\right.$$
$$+ \mu^{n-2}\{\tfrac{1}{4}(n-1)(n-2) G_n\}$$
$$\left. + \mu^{n+2}\left\{2Ae^\phi \frac{\partial G_n}{\partial \phi} - nAe^\phi G_n - A^2 \sin \psi_0 e^\phi F_n\right\}\right] \quad . . . . . . . . . . . . (10)$$

Expansions must now be assumed for $F_n$ and $G_n$, and the most general suitable expressions are given by

$$F_n = \Sigma\, a_{nm} e^{\frac{1}{2}m\phi}$$

and

$$G_n = \Sigma\, b_{nm} e^{\frac{1}{2}m\phi}.$$

The index $m$ is a positive or negative integer, $\Sigma$ denotes the sign of summation, and the $a$'s and $b$'s are coefficients to be determined subsequently.

When $n$ is an odd integer $m$ is also an odd integer, and when $n$ is an even integer which may include zero, $m$ is also an even integer which may include zero.

If now we insert the expansions for $F_n$ and $G_n$ in (10) and consider all the postulated values of $n$ and $m$, we have to select in the first place the coefficient of $\mu^n e^{\frac{1}{2}m\phi}$ in the expansion of the differential equation, and in the second place the coefficient of $\tan \frac{1}{2}(\psi - \psi_0)\,\mu^n e^{\frac{1}{2}m\phi}$.

These coefficients, along with all those obtained by giving the prescribed values to $n$ and $m$, must be zero in order to ensure the satisfaction of the wave equation.

Hence, writing

$$A_{nm} = \tfrac{1}{4}m^2 - \tfrac{1}{4}n^2 - \tfrac{1}{2}mA\cos\psi_0 - \tfrac{1}{4}A^2\sin^2\psi_0,$$

and selecting the aforesaid coefficients, we must have

$$a_{nm}A_{nm} + a_{n,\,m-2}\left(-\frac{m-2}{2} + n\right)A + b_{n,\,m-2}(-A^2\sin\psi_0)$$
$$+ a_{n+2,\,m}\,\tfrac{1}{4}(n+2)(n+1) + a_{n-2,\,m-2}(m-n)A + b_{n-2,\,m-2}(A^2\sin\psi_0) = 0, \quad (11)$$

and

$$b_{nm}A_{nm} + b_{n,\,m-2}\left(-\frac{m-2}{2} + n - 1\right)A$$
$$+ b_{n+2,\,n}\,\tfrac{1}{4}\,n(n+1) + b_{n-2,\,m-2}(m-n)A + a_{n-2,\,m-2}(-A^2\sin\psi_0) = 0. \quad (12)$$

When $\kappa a$ is small a first approximation to the solution of (9) is obtained as follows. The omission of the third term and the two last terms on the left-hand side of (9) is equivalent to writing

$$A_{nm} = \tfrac{1}{4}m^2 - \tfrac{1}{4}n^2,$$

and omitting the $b$'s in (11).

Under these circumstances, if we put $m = n$ in (11), the coefficients of $a_{nm}$ and $a_{n-2,\,m-2}$ vanish, thus indicating that the series obtained by retaining only the $a$'s whose suffixes are equal is a solution when $\kappa a$ is small.

The general solution will, of course, depend upon the form which we assign to $F_0$ or $F_1$. We must assume that $m$ is a positive integer, otherwise F would increase indefinitely within the space enclosed between the two planes. In the first solution, therefore, both $n$ and $m$ are assumed to be odd integers, and further it is postulated that $F_1$ contains only the single term $a_{11}e^{\frac{1}{2}\phi}$. It further will be assumed provisionally that $G_1$ contains only the single term $b_{11}e^{\frac{1}{2}\phi}$. We have, therefore, initially

$$a_{13} = a_{15} = a_{17} = \ldots = 0,$$

and

$$b_{13} = b_{15} = b_{17} = \ldots = 0.$$

Putting $n = 1$ and $m = 3$ in (11) and (12) there are obtained the equations

$$\tfrac{1}{2}Aa_{11} - A^2\sin\psi_0 b_{11} + \tfrac{1}{4}\,.\,3\,.\,2\,a_{33} = 0,$$

and

$$-\tfrac{1}{2}Ab_{11} + \tfrac{1}{4}\,.\,1\,.\,2\,b_{33} = 0.$$

Again, putting $n = 1$ and $m = 5$ in (11) and (12) we have

$$\tfrac{1}{4} \cdot 3 \cdot 2\, a_{35} = 0$$

and

$$\tfrac{1}{4} \cdot 1 \cdot 2\, b_{35} = 0.$$

Proceeding in this way it is readily seen that a solution can be obtained in which the $b$'s, whose suffixes are equal, are not required, and in which, further, all the $b$'s and $a$'s, containing a second suffix which is greater than the first, may be omitted.

Finally, if we put $m = 1$ in (12), we see that any $b$, whose second suffix is unity, may be omitted.

The solution we are seeking can now be divided up into series, and we consider, to begin with, that part of the solution consisting of the series which contains only those coefficients whose suffixes are equal. Putting, then, $m = n + 2$ in (11) it follows that

$$\tfrac{1}{4}(n+2)(n+1)\, a_{n+2,\, n+2} + \tfrac{1}{2} n \mathrm{A} a_{nn} = 0,$$

or

$$a_{n+2,\, n+2} = -\frac{2n}{(n+2)(n+1)} \mathrm{A} a_{nn}. \qquad (13)$$

Write now

$$a_{11} = \mathrm{A}^{\frac{1}{2}},$$

and

$$\mathrm{A}^{\frac{1}{2}} e^{\frac{1}{2}\phi} \cos \tfrac{1}{2}(\psi - \psi_0) = \alpha.$$

Denoting by $\mathrm{S}_0$ the series under consideration, it follows from (13) that

$$\mathrm{S}_0 = \alpha - \frac{\alpha^3}{1\,!\,3} + \frac{\alpha^5}{2\,!\,5} - \frac{\alpha^7}{3\,!\,7} + \ldots = \int_0^\alpha e^{-v^2}\, dv, \qquad (14)$$

which is the FRESNEL integral.

Next we consider that part of the solution which consists of the series containing only those $b$'s whose suffixes differ by two. Putting $m = n$ in (12), it follows that

$$n(n+1)\, b_{n+2,\, n} + 2n\mathrm{A} b_{n,\, n-2} - 4\mathrm{A}^2 \sin \psi_0 a_{n-2,\, n-2} = 0.$$

Writing $\mathrm{B} = 2\mathrm{A}$, we have in succession

$$-\mathrm{B}^2 \sin \psi_0 a_{11} + 3 \cdot 4 b_{53} + 0 + \ldots + 0 = 0,$$
$$-\mathrm{B}^2 \sin \psi_0 a_{33} + 5\mathrm{B} b_{53} + 5 \cdot 6 b_{75} + \ldots + 0 = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$-\mathrm{B}^2 \sin \psi_0 a_{n-2,\, n-2} + n(n+1)\, b_{n+2,\, n} + \ldots + n\mathrm{B} b_{n,\, n-2} = 0.$$

Eliminating all the $b$'s, except $b_{n+2,\,n}$, we form the determinant

$$0 = \begin{vmatrix} a_{11} & a_{33} & a_{55} & \cdots & a_{n-2,\,n-2} - \dfrac{n\,(n+1)\,b_{n+2,\,n}}{B^2 \sin\psi_0} \\ 3.4 & 5B & 0 & \cdots & 0 \\ 0 & 5.6 & 7B & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & nB \end{vmatrix}.$$

Now, by an easily proved property of determinants, the value of any minor in the above determinant with respect to the components of the first row, is simply the product of the components of the diagonal of the said minor.

Accordingly, upon inserting the equivalents of the $a$'s which have already been found, it follows that

$$(-1)^{\frac{1}{2}(n-3)} \frac{(n+1)!}{2B^2 \sin\psi_0} b_{n+2,\,n} = 2^{\frac{1}{2}(n-3)} A^{\frac{1}{2}(n-2)}\, 1 \,.\, 3 \,.\, 5 \ldots n$$
$$\times \left\{\frac{1}{1\,.\,3} + \frac{2}{3\,.\,5} + \frac{3}{5\,.\,7} + \ldots + \tfrac{1}{2}\,\frac{n-1}{n\,(n-2)}\right\}.$$

Now the expression in brackets is equivalent to

$$\tfrac{1}{4}\,\{2\Sigma_n - (n+1)/n\},$$

where

$$\Sigma_n = 1 + \tfrac{1}{3} + \tfrac{1}{5} + \ldots + 1/n.$$

Hence, finally,

$$b_{n+2,\,n} = (-1)^{\frac{1}{2}(n-3)} \sin\psi_0 A^{\frac{1}{2}(n+2)} \left\{\frac{1}{(\frac{1}{2}n+\frac{1}{2})!}\,\Sigma_n - \frac{1}{n}\,\frac{1}{(\frac{1}{2}n-\frac{1}{2})!}\right\}.$$

Denoting by $T_2$ the series under consideration, and putting $A^{\frac{1}{2}} e^{\frac{1}{2}i\phi} \cos\frac{1}{2}(\psi-\psi_0) = \alpha$ as before, it follows that

$$T_2 = \sin\psi_0 \tan\tfrac{1}{2}(\psi-\psi_0)\,\mu^2 A\left[-\alpha\,\frac{1}{1!}\,\Sigma_1 + \alpha^3\,\frac{1}{2!}\,\Sigma_3 - \alpha^5\,\frac{1}{3!}\,\Sigma_5 + \ldots\right.$$
$$\left. + \alpha\,\frac{1}{1}\,\frac{1}{0!} - \alpha^3\,\frac{1}{3}\,\frac{1}{1!} + \alpha^5\,\frac{1}{5}\,\frac{1}{2!} - \ldots\right],$$

$$= \sin\psi_0 \tan\tfrac{1}{2}(\psi-\psi_0)\,\mu^2 A\left[-\frac{1}{\alpha}\int_0^1 (e^{-\alpha^2 u^2} - e^{-\alpha^2})\,\frac{1}{1-u^2}\,du + \int_0^\alpha e^{-v^2}\,dv\right], \quad . \quad (15)$$

since

$$\Sigma_n = \int_0^1 \frac{1-u^{2\,\frac{1}{2}(n+1)}}{1-u^2}\,du.$$

We shall next consider the series obtained by putting $m = 1$ in (11). We have

$$(n+2)(n+1)\,a_{n+2,1} + (1 - n^2 - A^2 \sin^2 \psi_0 - 2A \cos \psi_0)\,a_{n,1} = 0. \quad . \quad (16)$$

Putting $n = 1$ in (16) we obtain

$$a_{31} = \frac{1}{3.2}(A^2 \sin^2 \psi_0 + 2A \cos \psi_0)\,A^{\frac{1}{2}}. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (17)$$

Hence the series which contains those $a$'s whose suffixes differ by two may be expressed in the form

$$\sin^2 \psi_0\, \mu^2 A^2 f_2(\alpha) + \cos \psi_0\, 2A\mu^2 g_2(\alpha),$$

where $f_2$ and $g_2$ are functions of $\alpha$.

Putting $n = 3$ in (16) we obtain

$$a_{51} = \frac{1}{5\,!}(A^2 \sin^2 \psi_0 + 2A \cos \psi_0)(A^2 \sin^2 \psi_0 + 2A \cos \psi_0 + 8)\,A^{\frac{1}{2}}. \ . \ . \quad (18)$$

Similarly, on putting $n = 5$ and $m = 3$ in (12) we obtain

$$b_{73} = \frac{1}{5.6}\left[(16 + A^2 \sin^2 \psi_0 + 6A \cos \psi_0) + \tfrac{1}{2}(A^2 \sin^2 \psi_0 + 2A \cos \psi_0)\right]\frac{A^{5/2} \sin \psi_0}{3}. \ (19)$$

We may conclude, by an inspection of these expressions and an extension to include the series which contain coefficients with suffixes that differ by 6, 8, etc., that when $2\kappa a \sin \phi_0/\pi$ is of the order unity or less and when $|\psi - \psi_0|$ does not differ much from $\pi$, that is in the neighbourhood of the geometrical shadow, the series $S_0$ is a first approximation to the solution under consideration and $S_0 + T_2$ a second.

When $2\kappa a \sin \phi_0/\pi$ is greater than unity it is necessary that $2\kappa a \sin \phi_0 \,.\, \mu/\pi$ should be small if the above approximation is to hold.

## *Second Solution.*

It is now necessary to consider a second solution of the differential equation (9), which is required in order to make the complete solution tend to vanish within the geometrical shadow.

We make the same assumptions as before except that in this case the indices $n$ and $m$ are even integers.

Putting $m = n$ in (12) we obtain

$$n(n+1)\,b_{n+2,n} + 2An b_{n,n-2} - 4A^2 \sin \psi_0 a_{n-2,n-2} = 0. \quad . \ . \ . \quad (20)$$

We must assume that the $b$'s, whose suffixes are equal, vanish; and that the $a$'s, whose suffixes are equal except $a_{00}$, vanish. Putting $n = 2$ in (20) we obtain

$$b_{42} = \frac{1}{3.2}\,(4\text{A}^2 \sin\psi_0 \,.\, a_{00} - 4\text{A}b_{20})$$

$$= \frac{4}{3.2}\,\text{AC, say.}$$

Putting $n = 2, 4, 6$, etc., in succession in (20), we have then

$$b_{42} = +4\,\frac{\text{A}}{3\,!}\,.\,\frac{2}{2}\,\text{C},$$

$$b_{64} = -4^2\,\frac{\text{A}^2}{5\,!}\,.\,\frac{4}{2}\,.\,\frac{2}{2}\,\text{C},$$

$$b_{86} = +4^3\,\frac{\text{A}^3}{7\,!}\,.\,\frac{6}{2}\,.\,\frac{4}{2}\,.\,\frac{2}{2}\,\text{C}, \quad \text{etc.}$$

For the series under consideration we shall write $\text{R}_2$. Then

$$\text{R}_2 = \tan\tfrac{1}{2}(\psi - \psi_0)\,\mu^2 \left\{\frac{2}{3}\alpha^2 - \frac{2\,.\,2}{3\,.\,5}\alpha^4 + \frac{2\,.\,2\,.\,2}{3\,.\,5\,.\,7}\alpha^6 - \dots\right\}\text{C}, \quad . \;.\;. \quad (21)$$

where $\alpha$ has the previous significance. This is the only series that will be required in this solution, when $\mu$ is small. The expression for $\text{R}_2$ is easily found to be

$$\text{R}_2 = \tan\tfrac{1}{2}(\psi - \psi_0)\,\mu^2 \left\{1 - \frac{1}{\alpha}\,e^{-\alpha^2}\int_0^\alpha e^{v^2}\,dv\right\}\text{C}. \quad . \;.\;.\;. \quad (22)$$

Collecting the foregoing results we have in the neighbourhood of the geometrical shadow, $\psi - \psi_0 = \pi$, where, if $\phi$ is large, $\alpha$ is not necessarily small, that is in the important region of the FRESNEL diffraction phenomena,

$$\text{F} = \text{B}\,(\text{S}_0 + \text{T}_2) + a_{00} + \text{R}_2, \quad . \;.\;.\;.\;.\;.\;.\;.\;. \quad (23)$$

where B is a constant.

Before putting F into its final form, let us consider the nature of the function $\text{T}_2$. Let

$$\text{T} = \int_0^1 (e^{-\alpha^2 u^2} - e^{-\alpha^2})\,\frac{1}{1 - u^2}\,du. \quad . \;.\;.\;.\;.\;.\;.\;.\;. \quad (24)$$

This expression is unsuitable, in its present form, for calculation. It must accordingly be transformed as follows.

From (24) we have

$$\text{T} + \frac{d\text{T}}{d\alpha^2} = \int_0^1 e^{-\alpha^2 u^2}\,du$$

$$= \frac{1}{\alpha}\int_0^\alpha e^{-v^2}\,dv$$

$$= \frac{1}{\alpha}\,e^{-\alpha^2}\left(\alpha + \frac{2}{3}\alpha^3 + \frac{2\,.\,2}{3\,.\,5}\alpha^5 + \dots\right).$$

Hence, upon integration,

$$T = \alpha^2 e^{-\alpha^2}\left(1 + \frac{1}{2}\cdot\frac{2}{3}\alpha^2 + \frac{1}{3}\cdot\frac{2\,.\,2}{3\,.\,5}\alpha^4 + \ldots\right)$$

$$= \alpha^2 e^{-\alpha^2}(M + iN), \quad \text{say.} \qquad (25)$$

These functions M and N can be calculated with ease to include the first maximum of FRESNEL'S diffraction phenomena. For this purpose it is convenient to write

$$\alpha^2 = i\tfrac{1}{2}\pi\,\beta^2,$$

so that

$$\beta = \frac{2}{\pi}(\kappa a \sin\phi_0)^{\frac{1}{2}} e^{\frac{1}{4}i\phi}\cos\tfrac{1}{2}(\psi - \psi_0). \qquad (26)$$

If, further, we write $\pi\beta^2 = \gamma$, we have

$$M = 1 - \frac{1}{3}\cdot\frac{1}{3\,.\,5}\gamma^2 + \frac{1}{5}\cdot\frac{1}{3\,.\,5\,.\,7\,.\,9}\gamma^4 \ldots,$$

and

$$N = \frac{1}{2}\cdot\frac{1}{3}\gamma - \frac{1}{4}\cdot\frac{1}{3\,.\,5\,.\,7}\gamma^3 + \ldots.$$

These series are rapidly convergent and have been used to calculate the following table, which has been found useful :—

| $\gamma$. | M. | N. |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 0·978 | 0·164 |
| 2 | 0·914 | 0·315 |
| 3 | 0·816 | 0·440 |
| 4 | 0·694 | 0·530 |
| 5 | 0·561 | 0·581 |
| 6 | 0·430 | 0·595 |
| 7 | 0·312 | 0·575 |
| 8 | 0·215 | 0·529 |
| 9 | 0·145 | 0·467 |

In order to determine the asymptotic value of T, we have already found that

$$T + \frac{dT}{d\alpha^2} = \frac{1}{\alpha}\int_0^\alpha e^{-v^2}\,dv.$$

Hence, when $\alpha$ is large,

$$T + \frac{dT}{d\alpha^2} \to 0,$$

or

$$T \to e^{-\alpha^2}.$$

We can now write down the required expression for F, on developing (23). It is obtained as follows.

Let $b_{20} = 0$, which must be so, since it is an independent coefficient. Then

$$C = A \sin \psi_0 \, . \, a_{00}.$$

Let, further,

$$\frac{a_{00}}{B} = \int_{-\infty}^{0} e^{-v^2}\, dv = \tfrac{1}{2}\pi^{\frac{1}{2}}.$$

Then

$$\frac{F}{B} = \{1 + \sin \psi_0 \tan \tfrac{1}{2}(\psi - \psi_0)\, \mu^2 A\} \int_{-\infty}^{\alpha} e^{-v^2}\, dv$$
$$+ \sin \psi_0 \tan \tfrac{1}{2}(\psi - \psi_0)\, \mu^2 A \left\{ -\frac{1}{\alpha} T - \frac{1}{2}\frac{\pi^{\frac{1}{2}}}{\alpha} e^{-\alpha^2} \int_0^{\alpha} e^{v^2}\, dv \right\}. \quad . \quad . \quad . \quad (27)$$

It should be noted that, since

$$\alpha = \left(2i\kappa \frac{a}{\pi} \sin \phi_0\right)^{\frac{1}{2}} e^{\frac{1}{2}\phi} \cos \tfrac{1}{2}(\psi - \psi_0),$$

the integrals in (27) can be put readily into FRESNEL'S form.

The solution for the reflected wave, which is obtained by changing the sign of $\psi_0$ in the solution already discussed, must be small in the neighbourhood of FRESNEL'S diffraction phenomena, and it will have no appreciable effect upon the rapid alterations of intensity in that neighbourhood.

Let us assume that the edges A and B (fig. 1) of the two semi-infinite planes are joined by a plane face, so that we have in effect a thick semi-infinite plane. It is then apparent that the waves, which would otherwise have entered the space between the planes, will now be scattered by the face AB. The effect of these scattered waves may confidently be assumed to be negligible in the region under consideration.

When we consider the problem of the diffraction of light by a straight edge, we picture the latter as the edge of a thin sheet of metal, ignoring the fact that the thickness of the sheet is probably great when expressed in terms of wave-length of the incident light. Observation and also theory, as we have just seen, indicate that the thickness does not essentially affect the phenomena of diffraction in the neighbourhood of the shadow at points sufficiently distant from the straight edge, that is at points where those phenomena are usually observed.

But there is one other important point to notice.

The geometrical shadow has so far been defined as the locus $\psi - \psi_0 = \pi$.

Let us consider perpendicular incidence, in which case $\psi_0 = \frac{1}{2}\pi$. Then $\psi, = \frac{3}{2}\pi$, defines the geometrical shadow. In this case let P (fig. 1) be a point upon the geometrical shadow. Then the co-ordinates of P are, if we refer $x$ and $y$ to C as origin,

$$x = -\frac{a}{\pi}(1 + \phi), \qquad y = -\frac{a}{\pi}(\tfrac{1}{2}\pi + e^{\phi}).$$

Join P to C, the centre of the edge, and let PC make an angle $\theta_0$ with the direction of incidence. Then

$$\tan\theta_0 = (1 + \phi)/(\tfrac{1}{2}\pi + e^{\phi}).$$

It is clear that, for large values of $\phi$, $\theta_0 \rightarrow$ zero, but the shadow is displaced by an amount

$$x, \quad = -a\,(1 + \phi)/\pi,$$

from what may be called the true geometrical shadow.

The insertion of the face AB is not essential to the problem, and in its absence the latter is completely solved, but the necessity of discussing the effect of thickness is clear in a study of what is probably the most celebrated problem in diffraction.

When $\psi_0 = 0$ the solutions are much simplified. This case is applied in the summary to a problem in sound waves.

### *The Motion within the Planes.*

The problem of the two planes has points of interest and importance in addition to those associated with FRESNEL'S theory. We shall apply the theory, so far developed, to points in the space between the planes. We may first point out that, if the amplitude of the undisturbed incident wave is unity, we must (see equation (27)) put $B = 1/\pi^{\frac{1}{2}}$.

Return now to the equation (16), which determines the coefficients of the series with $e^{\frac{1}{2}\phi}$ raised to the first power only. When $\phi$ is large and negative, that is well inside the two planes, this series need only be considered. The series is easily determined, but it takes a specially interesting form for perpendicular incidence, in which case $\psi_0 = \frac{1}{2}\pi$.

In this case, if we assume that

$$1 - n^2 - A^2 = 0,$$

the series terminates.

Let us take $n = 3$, so that

$$\kappa a \sin\phi_0 = \pi\sqrt{2}.$$

Then it follows that

$$F = \alpha + \frac{1}{3\,.\,2} A^2\mu^2\alpha.$$

Incorporating the effect of the reflected wave and considering the electromagnetic waves postulated, the complete expression inside the planes is

$$u_0 = \frac{1}{\pi^{\frac{1}{2}}\sqrt{2}}\,(A^{\frac{1}{2}} e^{-\frac{1}{2}\phi})\left\{2\sin\tfrac{1}{2}\psi + \frac{A^2}{6}(\tfrac{3}{2}\sin\tfrac{1}{2}\psi + \tfrac{1}{2}\sin\tfrac{3}{2}\psi)\right\} e^{i\kappa(z\cos\phi_0 + ct)}.$$

If the incident waves for any angle of incidence are waves of sound, and if $\kappa a$ be small, the amplitude at the mouth of the planes differs very little from unity, thus verifying the fundamental postulate in the theory of resonators that there is a loop at the mouth.

*The Flow of Energy. Third Solution.*

The problem of diffraction when $\kappa a$ is small is of interest, since it can be completely solved. The solution enables us to investigate the propagation of energy around the edge of a thick plane into the space within the geometrical shadow.

For this problem we shall assume that the thick semi-infinite plane (thick but thin compared with the wave-length) is bounded by the outer sides of the planes $\psi = 0$ and $\psi = 2\pi$ and the curved surface $\phi = 0$. Still another solution of equation (9) is required when $\kappa a$ is small, and in this case we write

$$F = a_{1,-1}\, e^{-\frac{1}{2}\phi}.$$

Inspection shows that it is still possible to obtain a solution in ascending odd powers of $e^{\frac{1}{2}\phi}$.

In equation (11), connecting the coefficients, we write $m = n$, and for that series in which the suffixes of the $a$'s differ by two we must have

$$2Aa_{n,n-2} + (n+1)\, a_{n+2,n} = 0.$$

This equation leads to

$$a_{31} = -Aa_{1,-1},$$

and, putting $a_{1,-1} = A^{\frac{1}{2}}$, the succeeding terms are obtained at once.

Let $\alpha$ have the same significance as hitherto, and, writing $S_{-12}$ for this series, we have

$$S_{-12} = A^{\frac{1}{2}}\, e^{-\frac{1}{2}\phi} \cos \tfrac{1}{2}(\psi - \psi_0)\, e^{-\alpha^2}. \qquad (28)$$

The subsequent series can be obtained by the processes already used, but $S_{-12}$ is the only series which need be retained when $\kappa a$ is small.

If the complete solution be now divided up into four parts $u_i$, $v_i$ containing $\cos \frac{1}{2}(\psi - \psi_0)$ and its powers, and $u_r$, $v_r$ containing $\cos \frac{1}{2}(\psi + \psi_0)$ and its powers, we can write down the solution we are seeking. Let it be for incident sound waves, in which case it is

$$u_0 = u_i + v_i + u_r + v_r.$$

where

$$u_i = \tfrac{1}{2}P + \frac{P}{\pi^{\frac{1}{2}}}\int_0^\alpha e^{-v^2}\, dv,$$

$$v_i = \frac{P}{\pi^{\frac{1}{2}}}\, A^{\frac{1}{2}}\, e^{-\frac{1}{2}\phi} \cos \tfrac{1}{2}(\psi - \psi_0)\,.\, e^{-v^2},$$

and in which

$$P = \exp\{\tfrac{1}{2}Ae^{\phi} \cos(\psi - \psi_0) + \tfrac{1}{2}A(1 - \phi)\cos\psi_0 + i\kappa z \cos\phi_0 + i\kappa ct\},$$

$$\alpha = A^{\frac{1}{2}} e^{\frac{1}{2}\phi} \cos \tfrac{1}{2}(\psi - \psi_0),$$

as hitherto.

The expressions for $u_r$ and $v_r$ are precisely similar but with the sign of $\psi_0$ changed. Now in the neighbourhood of the edge, $\phi = 0$, this expression for $u_0$ reduces approximately to

$$u \,.\, e^{i\kappa(z\cos\phi_0 + ct)},$$

where

$$u = 1 + \frac{1}{\pi^{\frac{1}{2}}} A^{\frac{1}{2}} \cosh \tfrac{1}{2}\phi \cos \tfrac{1}{2}(\psi - \psi_0) + \frac{1}{\pi^{\frac{1}{2}}} A^{\frac{1}{2}} \cosh \tfrac{1}{2}\phi \cos \tfrac{1}{2}(\psi + \psi_0). \quad . \quad . \quad (29)$$

If we differentiate this expression with respect to $\phi$, the derivative vanishes when $\phi = 0$.

Now the rate of variation of $u$ along the normal to the curve $\phi =$ const. at any point is denoted by $\partial u/\partial n$, or by

$$\frac{\partial u}{\partial \phi} \cdot \frac{\partial \phi}{\partial n}.$$

But

$$\frac{\partial \phi}{\partial n} = \frac{\pi}{a}(1 - 2e^{\phi}\cos\psi + e^{2\phi})^{-\frac{1}{2}}.$$

Considering, therefore, the second term of (29),

$$\frac{\partial u}{\partial n} = \left(\frac{A}{\pi}\right)^{\frac{1}{2}} \cos \tfrac{1}{2}(\psi - \psi_0) \frac{\pi}{2a} \frac{\sinh \frac{1}{2}\phi}{(1 - 2e^{\phi}\cos\psi + e^{2\phi})^{\frac{1}{2}}}.$$

When $\phi$ is small this reduces to

$$\frac{\partial u}{\partial n} = \left(\frac{A}{\pi}\right)^{\frac{1}{2}} \cos \tfrac{1}{2}(\psi - \psi_0) \frac{\pi}{2a} \cdot \frac{\sinh \frac{1}{2}\phi}{2 \sin \frac{1}{2}\psi},$$

and, in the neighbourhood of the point A (fig. 1), where $\psi$ is also small, it reduces to

$$\frac{\partial u}{\partial n} = \left(\frac{A}{\pi}\right)^{\frac{1}{2}} \frac{\pi}{2a} \cos \tfrac{1}{2}\psi_0 \cdot \tfrac{1}{2} \cdot \frac{\phi}{\psi}.$$

This is, at first sight, indeterminate in the limit when both $\phi$ and $\psi$ are zero. But, if we write $G = \partial u/\partial n$, it is clear that $\partial G/\partial \psi = 0$ at all points of the curve $\phi = 0$. Hence, if G be also zero at all points of the same curve, it must be zero when both $\phi = 0$ and $\psi = 0$.

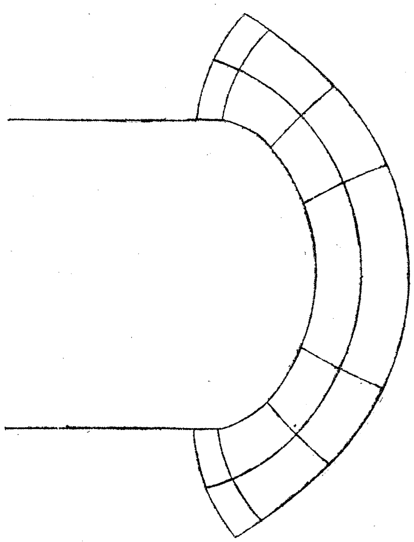

FIG. 2.

The same reasoning applies to the last term of (29).

In fig. 2 the nature of the motion near the edge is indicated. The lines radiating from the edge are the curves of constant phase in the case of sound waves. The lines which resemble the family $\phi =$ const. are the curves of constant phase in the case of the prescribed electromagnetic waves.

We shall now examine the nature of the flow of energy into the space within the geometrical shadow in the case of the electromagnetic waves. To this end, neglecting the reflected wave and the wave scattered from the end $\phi = 0$ which are negligible far inside the shadow, we require only

$$u_i = \frac{P}{\pi^{\frac{1}{2}}} \int_{-\infty}^{\alpha} e^{-v^2}\, dv.$$

Now when $\cos \frac{1}{2}(\psi - \psi_0)$ is negative the asymptotic value of $u_i \propto Pe^{-\alpha^2}/\alpha$.

Since $\kappa a$ is small, denoting the perpendicular distance of a point well within the shadow from the axis of $z$ by $r$, where $r$ is great compared with $a$; then, if $\phi_1 = \pi - \phi_0$,

$$u_i\,(\text{asymp.}) \propto \frac{1}{\alpha}\, e^{-i\kappa(z\cos\phi_1 + r\sin\phi_1) + i\kappa ct} \quad \ldots \ldots \ldots \quad (30)$$

Let Q be this point, distant R from O and let QN $(= r)$ be the perpendicular upon the axis of $z$.

On applying equations (1) to (30) in order to determine the forces, it is found that the most important terms arise from the differentiation of the argument

$$z\cos\phi_1 + r\sin\phi_1 = R.$$

Since this argument is independent of $\psi$ the electric force lies in the plane OQN and the magnetic force at Q is perpendicular to this plane.

Now the most important part of

$$\frac{1}{c^2}\frac{\partial Z}{\partial t} \propto -i\kappa u_i\,(\text{asymp.}) \text{ and that of } \frac{\partial^2 R}{\partial z^2} \propto -i\kappa u_i\,(\text{asymp.})\,\frac{\sin^2\phi_1}{R};$$

also the most important part of

$$\frac{1}{c^2}\frac{\partial X}{\partial t} \propto -i\kappa u_i\,(\text{asymp.}) \text{ and that of } \frac{\partial^2 R}{\partial r\,.\,\partial z} \propto +i\kappa u_i\,(\text{asymp.})\,\frac{\sin\phi_1\cos\phi_1}{R}.$$

It is clear, then, that the resultant electric force lies in the plane OQN and is perpendicular to OQ. It follows that, if the surface of a cone, with apex at O and axis of Z as axis, be imagined, then at points on this surface well within the shadow energy is transmitted into the shadow along the generating lines of the cone. The origin O is, of course, arbitrary.

## *Symmetrical Problems.*

The solutions are simplified and physically important, when there is symmetry about the plane of $xz$.

When the wave enters directly into the space between the two planes we may put $\psi_0 = \pi$. In this case equation (11) among the coefficients, omitting the third and sixth terms on the left-hand side, is alone required.

Employing a process similar to that used in the determination of the coefficients of the series $T_2$, it is found that

$$(-1)^{\frac{1}{2}(n-1)}\left(\tfrac{1}{2}n+\tfrac{1}{2}\right)!\,a_{n+2,\,n} = -A^{\frac{1}{2}(n+2)}\left(\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{n+2}\right). \quad . \quad . \qquad (31)$$

Denoting the corresponding series by $S_2$, we have

$$S_2 = -\cos^2\tfrac{1}{2}(\psi-\pi)\frac{A}{\alpha}\int_0^1\left(e^{-\alpha^2u^2}-e^{-\alpha^2}\right)\frac{u^2}{1-u^2}\,du. \quad . \quad . \quad . \quad . \quad . \qquad (32)$$

This expression can be treated in precisely the same manner as we have treated the expression for $T_2$.

As typical of the processes required for a complete solution we shall consider, finally, that series in which the suffixes of the $a$'s differ by four.

From (11) we have

$$-2Aa_{n-4,\,n-6}+a_{n-2,\,n-4}A_{n-2,\,n-4}+a_{n-2,\,n-6}\tfrac{1}{2}(n+2)A+\tfrac{1}{4}n(n-1)a_{n,\,n-4}=0. \qquad (33)$$

In (33) write $n = 5$, 7, etc., in succession, and put

$$B_{n-4} = -2Aa_{n-4,\,n-6}+A_{n-2,\,n-4}\,a_{n-2,\,n-4},$$

so that the B's are known from the series for $S_2$. It follows then, remembering that no term with a negative suffix exists, that

$$0 = \begin{vmatrix} B_1 & B_3 & B_5 & B_7 & \ldots & \{B_{n-4}+\tfrac{1}{4}n(n-1)a_{n,\,n-4}\} \\ \tfrac{1}{4}.5.4 & \tfrac{9}{2}A & 0 & 0 & \ldots & 0 \\ 0 & \tfrac{1}{4}.7.6 & \tfrac{11}{2}A & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & \tfrac{1}{2}(n+2)A \end{vmatrix}. \qquad (34)$$

This determinant is obtained by precisely the same procedure as before.

Let now the minors, representing the coefficients, of $B_1$, $B_3$, etc., be denoted by

$$\kappa_1A^{\frac{1}{2}(n-5)}, \quad \kappa_3A^{\frac{1}{2}(n-7)}, \text{ etc., respectively,}$$

so that the $\kappa$'s are purely numerical coefficients. Then

$$B_1\kappa_1A^{\frac{1}{2}(n-5)}-B_3\kappa_3A^{\frac{1}{2}(n-7)}+\ldots(-1)^{\frac{1}{2}(n-5)}\{B_{n-4}\kappa_{n-4}+\tfrac{1}{4}n(n-1)a_{n,\,n-4}\kappa_{n-4}\}=0. \quad (35)$$

Now we obtain easily from the series for $S_2$, if we write

$$\Sigma_n = \tfrac{1}{3} + \tfrac{1}{5} + \ldots + \frac{1}{n},$$

$$B_n = (-1)^{\frac{1}{2}(n+3)} \frac{1}{(\frac{1}{2}n+\frac{1}{2})!} \frac{n+1}{n+2} A^{\frac{1}{2}(n+2)} + (-1)^{\frac{1}{2}(n+1)} \frac{1}{(\frac{1}{2}n+\frac{1}{2})!} \cdot \frac{n}{2} A^{\frac{1}{2}(n+4)} \Sigma_{n+2}. \quad (36)$$

Inserting the values of the successive B's in (35), it follows that

$$A^{\frac{1}{2}(n-2)}\left(\frac{1}{1!}\cdot\frac{2}{3}\kappa_1 + \frac{1}{2!}\cdot\frac{4}{5}\kappa_3 + \ldots + \frac{1}{(\frac{1}{2}n-\frac{3}{2})!}\cdot\frac{n-3}{n-2}\cdot\kappa_{n-4}\right)$$
$$-\tfrac{1}{2}A^{\frac{1}{2}n}\left(\frac{1}{1!}\Sigma_3\kappa_1 + \frac{1}{2!}3\Sigma_5\kappa_3 + \ldots + \frac{1}{(\frac{1}{2}n-\frac{3}{2})!}\cdot\overline{n-4}\,\Sigma_{n-2}\kappa_{n-4}\right)$$
$$+(-1)^{\frac{1}{2}(n-5)}\cdot\tfrac{1}{4}n(n-1)\kappa_{n-4}a_{n,n-4} = 0. \quad (37)$$

Considering the values of the $\kappa$'s more closely we have

$$\frac{\kappa_1}{\kappa_{n-4}} = \frac{n(n+2)}{5\,.\,7}\cdot\frac{1!}{(\frac{1}{2}n-\frac{3}{2})!},$$
$$\frac{\kappa_3}{\kappa_{n-4}} = \frac{n(n+2)}{7\,.\,9}\cdot\frac{2!}{(\frac{1}{2}n-\frac{3}{2})!}, \text{ etc.}$$

Hence, when divided by $\kappa_{n-4}$, the coefficient of $A^{\frac{1}{2}(n-2)}$ in (37) becomes

$$\frac{2n(n+2)}{(\frac{1}{2}n-\frac{3}{2})!}\left\{\frac{1}{3\,.\,5\,.\,7} + \frac{2}{5\,.\,7\,.\,9} + \ldots + \frac{n-3}{2(n-2)n(n+2)}\right\} \quad (38)$$

The series within brackets in this expression can be summed, for

$$\tfrac{1}{2}\frac{n-3}{(n-2)n(n+2)} = \frac{3}{8}\cdot\frac{1}{n} - \frac{5}{16}\cdot\frac{1}{n+2} - \frac{1}{16}\cdot\frac{1}{n-2}.$$

The sum is

$$\frac{1}{24} - \frac{5}{16}\cdot\frac{1}{n+2} + \frac{1}{16}\cdot\frac{1}{n},$$

and the whole expression (38) reduces to

$$\frac{1}{(\frac{1}{2}n-\frac{3}{2})!}\cdot\tfrac{1}{12}(n-3)(n-1).$$

Denoting the numerical coefficient of $A^{\frac{1}{2}n}$ in (37) by $C_n$, that equation becomes

$$A^{\frac{1}{2}(n-2)}\frac{1}{(\frac{1}{2}n-\frac{3}{2})!}\cdot\tfrac{1}{12}(n-3)(n-1) - A^{\frac{1}{2}n}C_n + (-1)^{\frac{1}{2}(n-5)}\cdot\tfrac{1}{4}n(n-1)a_{n,n-4} = 0.$$

We can now write, for the series in which the suffixes of the $a$'s differ by four,

$$S_{24} - S_{14},$$

in which

$$S_{14} = A\frac{\mu^4}{3}\left(\alpha \cdot \frac{2}{5} - \frac{\alpha^3}{2\,!} \cdot \frac{4}{7} + \frac{\alpha^5}{3\,!} \cdot \frac{6}{9} - \dots\right),$$

$$= \frac{2}{3} \cdot \frac{A\mu^4}{\alpha^4}\int_0^\alpha u^4 e^{-u^2}\,du,$$

$$= \tfrac{1}{2}\frac{A\mu^4}{\alpha^4} S_0 - A\mu^4\left(\frac{1}{2\alpha^3} + \frac{1}{3\alpha}\right)e^{-\alpha^2}, \quad \dots\dots\dots\dots \quad (39)$$

and

$$S_{24} = A^2\mu^4 f(\alpha).$$

We shall not develop the function $f(\alpha)$ explicitly, for sufficient indications have now been given as to the derivation of each succeeding series. The results obtained are valid over a considerable region, and for a reasonably wide range of values of the ratio $\kappa a$.

One more series will be obtained, however, in order to complete the approximate solution for the space between the planes.

It is the series deduced by retaining only the first power of $e^{\frac{1}{2}\phi}$. In this series the general equation to be satisfied among the coefficients is

$$4a_{n1}A_{n1} + (n+2)(n+1)\,a_{n+2,\,1} = 0,$$

where

$$4A_{n1} = 1 - n^2 + 2A.$$

The development of the series is straightforward and, if it be denoted by $S_1$,

$$S_1 = A^{\frac{1}{2}} e^{\frac{1}{2}\phi}\mu\left\{1 - \frac{2A\mu^2}{3\,!} + \frac{2^2 A\,(A - 2\,.\,1\,.\,2)\,\mu^4}{5\,!} - \dots\right\}.$$

The series within brackets can easily be shown to be convergent for all values of $\kappa a$ and for all values of $\mu$ lying between 0 and 1. The series in which the suffixes of the $a$'s differ by 6, 8, etc., contain respectively $\mu^6$, $\mu^8$, etc., as factors, and, therefore, need not be retained when the value of $\mu$ is moderately small.

The approximate solution in the space between the planes, when $\phi$ is large and negative, is

$$\frac{2}{\pi^{\frac{1}{2}}} S_1\, e^{i\kappa(-x\sin\phi_0 + z\cos\phi_0 + ct)},$$

$-\frac{1}{2}\phi$ being, of course, written for $+\frac{1}{2}\phi$.

When $\kappa a$ is small this expression reduces to

$$\frac{2}{\pi^{\frac{1}{2}}} A^{\frac{1}{2}} e^{-\frac{1}{2}\phi}\mu\, e^{i\kappa(-x\sin\phi_0 + z\cos\phi_0 + ct)}. \quad \dots\dots\dots\dots \quad (40)$$

which can be expressed in Cartesian co-ordinates by means of the equations of the fundamental transformation.

It should be noted that, since A is purely imaginary, the factor $A^{\frac{1}{4}}$ in (40) implies the presence of $e^{i\frac{1}{8}\pi}$, so that there is a change of phase of one-eighth of a wave-length. The above results show that, in the case of the electromagnetic waves postulated, the wave dies out altogether as it proceeds into the space between the two planes when the latter are perfect conductors.

In the case of sound waves, however, the wave proceeds into the space between the two planes unaffected.

## *Efflux of Sound Waves.*

If the sound waves are emerging from the space contained between the two planes, it is necessary to use the form (5) of the differential equation.

Putting, then,

$$\gamma = a/\pi,$$

$$l = \sin \phi_0,$$

$$m = 0,$$

and

$$n = \cos \phi_0,$$

in that equation, it becomes, since

$$\frac{1}{q^2} = \frac{a^2}{\pi^2}(1 + e^{2\phi} - 2\,e^{\phi}\cos\psi),$$

$$\frac{\partial^2 F}{\partial \phi^2} + \frac{\partial^2 F}{\partial \psi^2} + A\,\frac{\partial F}{\partial \phi} - \tfrac{1}{4}A^2\,(e^{2\phi} - 2\,e^{\phi}\cos\psi)\,F = 0. \quad \text{(41)}$$

If we write $\pi - \psi$ for $\psi$ and expand in powers of $\cos\frac{1}{2}\psi$, this problem can be treated generally by methods similar to the foregoing.

But a first approximation, when $\kappa a$ is small, to the solution of (41) is the appropriate solution of the equation

$$\frac{\partial^2 F}{\partial \phi^2} + \frac{\kappa^2 a^2}{\pi^2}\sin^2\phi_0\,e^{2\phi}F = 0. \quad \text{(42)}$$

The solution required is, accordingly,

$$F = J_0\left(\frac{\kappa a}{\pi}\sin\phi_0\,e^{\phi}\right),$$

where $J_0$ is BESSEL's function of zero order. For the complete solution we have, therefore,

$$u = BJ_0\left(\frac{\kappa a}{\pi}\sin\phi_0 e^{\phi}\right)\exp i\kappa\left(\frac{a}{\pi}\sin\phi_0\,.\,\phi + z\cos\phi_0 + ct\right).$$

When $\phi$ becomes negative, that is in the interior space, $J_0$ approaches the value unity so that in this space there is a plane wave emerging into the outside air.

This problem is of some interest in connection with the theory of the motion of the air at the mouth of a resonator. When $\kappa a$ is small the only complete solution of the resonator problem hitherto given is that by RAYLEIGH for the transmission of plane waves through small apertures in plane screens.

## SECTION 3.—THE ELLIPTIC TRANSFORMATION.

As in the previous section there are some problems under this heading in which the advantages of expanding in powers of $\cos \frac{1}{2} (\psi - \psi_0)$ are very great. The problems referred to contain in their solution, so far as the function $\psi$ is concerned, powers of $\cos \frac{1}{2} (\psi - \psi_0)$ only, and, consequently, approximate solutions are very readily obtained in those regions throughout which $\cos \frac{1}{2} (\psi - \psi_0)$ is small.

The importance of the solutions lies in the analytical investigations which may be carried out in a very important region and which, moreover, are not necessarily restricted to small values of the ratio of the dimensions of the obstacle to the wave-length.

The complete extension of the solutions is straightforward but apparently laborious. It is not impossible, however, that mathematical investigation into the nature of the general equation which connects the coefficients may result in a workable method.

The transformation which we shall have under discussion in this section is the elliptic one

$$x + iy = a \sinh (\phi + i\psi),$$

so that

$$x = a \sinh \phi \cos \psi$$

and

$$y = a \cosh \phi \sin \psi.$$

Applying the equation (4) we have, upon putting

$$2i\kappa a \sin \phi_0 = A,$$

and re-arranging the terms,

$$\frac{\partial^2 F_0}{\partial \phi^2} + \frac{\partial^2 F_0}{\partial \psi^2} + A \frac{\partial F_0}{\partial \phi} \sinh \phi \cos (\psi - \psi_0) - A \frac{\partial F_0}{\partial \psi} \cosh \phi \sin (\psi - \psi_0)$$
$$+ A \frac{\partial F_0}{\partial \phi} \cos \psi_0 \cos \psi e^{-\phi} + A \frac{\partial F_0}{\partial \psi} \cos \psi_0 \sin \psi \, e^{-\phi} = 0. \quad . \quad . \quad (43)$$

We shall commence with a discussion of the case of incidence perpendicular to the axis of $x$. In this case $\psi_0 = \frac{1}{2}\pi$, and (43) reduces to

$$\frac{\partial^2 F_0}{\partial \phi^2} + \frac{\partial^2 F_0}{\partial \psi^2} + A \frac{\partial F_0}{\partial \phi} \sinh \phi \cos (\psi - \tfrac{1}{2}\pi) - A \frac{\partial F_0}{\partial \psi} \cosh \phi \sin (\psi - \tfrac{1}{2}\pi) = 0. \quad . \quad (44)$$

The solution of the problem for the incident wave will then be

$$u_i = e^{i\kappa(y \sin\phi_0 + z\cos\phi_0 + ct)} \,.\, F_0. \qquad (45)$$

In fig. 3 a semi-infinite plane OC has a lamina AB, whose centre line coincides with the edge of the semi-infinite plane, attached at right angles to it. Waves are incident in a direction perpendicular to OC, the axis of $x$.

We have to consider the possibility of a solution of (44) in powers of $\cos\frac{1}{2}(\psi - \frac{1}{2}\pi)$. Putting $\cos\frac{1}{2}(\psi - \frac{1}{2}\pi) = \mu$ we have, as before,

$$\left.\begin{aligned} \sin(\psi - \tfrac{1}{2}\pi)\frac{d}{d\psi}(\mu^n) &= -n\mu^n + n\mu^{n+2}, \\ \cos(\psi - \tfrac{1}{2}\pi)\,\mu^n &= 2\mu^{n+2} - \mu^n, \\ \frac{d^2}{d\psi^2}(\mu^n) &= \tfrac{1}{4}n(n-1)\,\mu^{n-2} - \tfrac{1}{4}n^2\mu^n. \end{aligned}\right\} \qquad (46)$$

The function $F_0$ must now be expressed in the form

$$F_0 = F_1\mu + F_3\mu^3 + F_5\mu^5 + \ldots + F_n\mu^n + \ldots \qquad (47)$$

Applying the differential equation (44) to the term $F_n\mu^n$ in the expansion of $F_0$, there is obtained the expression

$$\begin{aligned} \mu^n&\left\{\frac{\partial^2 F_n}{\partial\phi^2} - \tfrac{1}{4}n^2 F_n - A\frac{\partial F_n}{\partial\phi}\sinh\phi + AF_n n\cosh\phi\right\} \\ &+ \mu^{n-2}\left\{\tfrac{1}{4}n(n-1)F_n\right\} \\ &+ \mu^{n+2}\left\{2A\frac{\partial F_n}{\partial\phi}\sinh\phi - AF_n n\cosh\phi\right\}. \end{aligned} \qquad (48)$$

### *Incidence Perpendicular to x-axis. First Solution.*

In the case of the prescribed electromagnetic waves $F_n$ may be expanded in the form

$$F_n = \sum_{m=1}^{m=n} a_{nm}\sinh\tfrac{1}{2}m\phi.$$

In the case of sound waves $F_n$ may be expanded in the form

$$F_n = \sum_{m=1}^{m=n} a_{nm}\cosh\tfrac{1}{2}m\phi.$$

We shall consider the case of sound waves in detail.

Now

$$\cosh\tfrac{1}{2}m\phi\cosh\phi = \tfrac{1}{2}\left(\cosh\frac{m+2}{2}\phi + \cosh\frac{m-2}{2}\phi\right)$$

and

$$\sinh \tfrac{1}{2}m\phi \sinh \phi = \tfrac{1}{2}\left(\cosh \frac{m+2}{2}\phi - \cosh \frac{m-2}{2}\phi\right).$$

Hence, if we consider the single term of $F_n$, which $\propto \cosh \frac{1}{2}m\phi$, and substitute it in (48), we obtain the expression

$$\mu^n \left\{(\tfrac{1}{4}m^2 - \tfrac{1}{4}n^2)\cosh \tfrac{1}{2}m\phi - A\frac{m}{4}(\cosh \tfrac{1}{2}\overline{m+2}\phi - \cosh \tfrac{1}{2}\overline{m-2}\phi) + A\frac{n}{2}(\cosh \tfrac{1}{2}\overline{m+2}\phi + \cosh \tfrac{1}{2}\overline{m-2}\phi)\right\}$$
$$+ \mu^{n-2}\{\tfrac{1}{4}n(n-1)\cosh \tfrac{1}{2}m\phi\}$$
$$+ \mu^{n+2}\left\{A\frac{m}{2}(\cosh \tfrac{1}{2}\overline{m+2}\phi - \cosh \tfrac{1}{2}\overline{m-2}\phi) - A\frac{n}{2}(\cosh \tfrac{1}{2}\overline{m+2}\phi + \cosh \tfrac{1}{2}\overline{m-2}\phi)\right\}.$$

We can now select the coefficient of $\mu^n \cosh \frac{1}{2}m\phi$ in the expression obtained by substituting $F_0$ in the differential equation. Since this coefficient must be zero, we have

$$-\tfrac{1}{4}(n^2 - m^2)a_{nm} + \tfrac{1}{2}A(n - \tfrac{1}{2}m + 1)a_{n,m-2} + \tfrac{1}{2}A(n + \tfrac{1}{2}m + 1)a_{n,m+2}$$
$$- \tfrac{1}{2}A(n-m)a_{n-2,m-2} - \tfrac{1}{2}A(n+m)a_{n-2,m+2} + \tfrac{1}{4}(n+2)(n+1)a_{n+2,m} = 0. \quad (49)$$

We must now investigate, as far as possible, the nature of the coefficients derived from equation (49).

If we put $m = n + 2$ in (49) then, on account of the assumptions made in the expansions of the F's, we must have

$$Ana_{nn} + (n+2)(n+1)a_{(n+2)(n+2)} = 0.$$

Hence, for the series whose coefficients contain equal suffixes, we have, putting

$$a_{11} = (\tfrac{1}{2}A)^{\frac{1}{2}},$$
$$a_{33} = -\frac{1}{1\,!\,3}(\tfrac{1}{2}A)^{\frac{3}{2}},$$
$$a_{55} = \frac{1}{2\,!\,5}(\tfrac{1}{2}A)^{\frac{5}{2}},$$

etc., *i.e.*, the coefficients of the expansion of the FRESNEL integral.

If $S_0$ denote this series and if we write

$$(\tfrac{1}{2}A)^{\frac{1}{2}}e^{\frac{1}{2}\phi}\mu = \alpha \quad \text{and} \quad (\tfrac{1}{2}A)^{\frac{1}{2}}e^{-\frac{1}{2}\phi}\mu = \beta,$$

the series becomes

$$2S_0 = \int_0^\alpha e^{-u^2}du + \int_0^\beta e^{-u^2}du. \quad (50)$$

Since $a_{31}$ could form the first coefficient of an independent series, all those coefficients whose suffixes differ by two are not required. Again put $m = n - 2$ in (49), and it follows that

$$A(n+4)\,a_{n,\,n-4} + A\,.\,3na_{nn} + (n+2)(n+1)\,a_{n+2,\,n-2} = 0. \quad . \quad . \quad . \quad (51)$$

From (51) we can write down in succession

$$A3\,.\,3a_{33} + 5\,.\,4a_{51} + \quad 0 \quad + \ldots + \quad 0 \quad = 0,$$

$$A3\,.\,5a_{55} + 9Aa_{51} + 7\,.\,6a_{73} + \ldots + \quad 0 \quad = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A3\,.\,na_{nn} + \quad 0 \quad + \quad 0 \quad + \ldots + (n+4)\,Aa_{n,\,n-4} + (n+2)(n+1)\,a_{n+2,\,n-2} = 0.$$

From these equations we can eliminate all those coefficients, whose suffixes differ by four, except $a_{n+2,\,n-2}$, by means of the determinant

$$0 = \begin{vmatrix} 3a_{33} & 5a_{55} & 7a_{77} & (n-2)\,a_{n-2,\,n-2} & na_{nn} + \dfrac{(n+2)(n+1)\,a_{n+2,\,n-2}}{3A} \\ 5\,.\,4 & 9A & 0 & 0 & 0 \\ 0 & 7\,.\,6 & 11A & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & n(n-1) & (n+4)\,A \end{vmatrix}.$$

Inserting the values of the $a$'s with equal suffixes which we have already found, this determinant leads to

$$(-1)^{\frac{1}{2}(n-3)}\frac{(n+2)!}{3A\,.\,3\,.\,2}\,a_{n+2,\,n-2}$$
$$= (\tfrac{1}{2})^{3/2}\,A^{\frac{1}{2}n}\,5\,.\,7\ldots(n+4)\left\{\frac{1}{5\,.\,7} + \frac{1}{7\,.\,9} + \ldots + \frac{1}{(n+2)(n+4)}\right\}.$$

Now the series in brackets on the right-hand side of this equation can be summed and its value is

$$\frac{1}{10} \quad \frac{n-1}{n+4}.$$

Hence, finally,

$$a_{n+2,\,n-2} = (-1)^{\frac{1}{2}(n-3)}\,\tfrac{3}{5}\,(\tfrac{1}{2}A)^{\frac{1}{2}(n+2)}\left\{\frac{1}{(\frac{1}{2}n-\frac{1}{2})!} - \frac{1}{(\frac{1}{2}n+\frac{1}{2})!}\right\}. \quad . \quad . \quad . \quad (52)$$

Denoting the series under consideration by $S_4$ we have

$$2S_4 = \tfrac{3}{5}\,(\tfrac{1}{2}A)^2\,\mu^4\left[\frac{1}{\alpha}\left\{-e^{-\alpha^2} + \frac{1}{\alpha^2}(1-e^{-\alpha^2})\right\} + \frac{1}{\beta}\left\{-e^{-\beta^2} + \frac{1}{\beta^2}(1-e^{-\beta^2})\right\}\right]. \quad (53)$$

Since $\alpha^2$ is purely imaginary, $S_4$ vanishes for large values of $\alpha$ to the order $\alpha^{-1}$.





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For the subsequent series the factors involve $\mu^6$ and higher powers, and they also involve the second power of $\kappa a$ at the lowest. The solution of the problem for the incident wave is, accordingly, as far as we have taken it.

$$\frac{1}{\pi^{\frac{1}{2}}} e^{i\kappa(y \sin\phi_0 + z\cos\phi_0 + ct)} \left\{ \int_{-\infty}^{\alpha} e^{-u^2} du + \int_0^{\beta} e^{-u^2} du + 2S_4 \right\}. \quad \quad (54)$$

The corresponding solution for the reflected wave is

$$\frac{1}{\pi^{\frac{1}{2}}} e^{i\kappa(-y \sin\phi_0 + z\cos\phi_0 + ct)} \left\{ \int_{-\infty}^{\alpha'} e^{-u^2} du + \int_0^{\beta'} e^{-u^2} du + S'_4 \right\}, \quad \quad (55)$$

in which $\alpha'$, $\beta'$, and $S'_4$ differ from $\alpha$, $\beta$, and $S_4$ only in having $\psi + \frac{1}{2}\pi$ written in place of $(\psi - \frac{1}{2}\pi)$. If $\kappa a$ be not too great, the remarkably small effect, which the lamina has upon the diffraction phenomena in FRESNEL's region, will be observed.

### *Incidence Perpendicular to the x Axis. Second Solution.*

We shall now consider the problem of diffraction under the elliptic transformation by an alternative treatment of the case just discussed.

For this alternative solution of the problem of sound waves we expand $F_n$ in powers of $\cosh \frac{1}{2}\phi$.

We write, therefore,

$$F_n = \sum_{m=1}^{m=n} a_{nm} \cosh^m \tfrac{1}{2}\phi,$$

and we obtain for the general coefficient, which arises from the substituting of $F_0$ in the differential equation (44),

$$(\tfrac{1}{4}m^2 - \tfrac{1}{4}n^2 + Am - An)\, a_{nm} + \tfrac{1}{4}(n+2)(n+1)\, a_{n+2,\, m} - A(2m-n)\, a_{n-2,\, m}$$
$$- \tfrac{1}{4}(m+2)(m+1)\, a_{n,\, m+2} + A(2n-m+2)\, a_{n,\, m-2} + 2A(m-n)\, a_{n-2,\, m-2}. \quad (56)$$

This must be equated to zero.

Writing $m = n+2$ in (56), the general equation, which determines the series with coefficients whose suffixes are equal, is

$$(n+2)(n+1)\, a_{n+2,\, n+2} + 4An a_{nn} = 0.$$

Putting $a_{11} = (2A)^{\frac{1}{2}}$, this leads to,

$$a_{nn} = (-1)^{\frac{1}{2}(n-1)} \frac{(2A)^{\frac{1}{2}n}}{n \cdot (\frac{1}{2}n - \frac{1}{2})\,!}.$$

This series is again, therefore, the FRESNEL integral. Putting $m = n$ in (56) it is found that, on account of the vanishing of the first and last terms in that expression, the series with coefficients whose suffixes differ by two is independent and therefore not required.

Putting $m = n - 2$ in (56), then, since

$$n(n-1)a_{nn} + 4A(n-2)a_{n-2,\,n-2} = 0,$$

the expression, which determines the series with coefficients whose suffixes differ by four, reduces to

$$(n+2)(n+1)a_{n+2,\,n-2} + 8Aa_{n-2,\,n-2} + 4A(n+4)a_{n,\,n-4} = 0.$$

This leads to the determinant

$$0 = \left|\begin{array}{ccc:cc} a_{11} & a_{33} & a_{55} & a_{n-4,\,n-4} & \left\{a_{n-2,\,n-2} + \dfrac{(n+2)(n+1)a_{n+2,\,n-2}}{8A}\right\} \\ 5.4 & 4A.9 & 0 & 0 & 0 \\ 0 & 7.6 & 4A.11 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & n(n-1) & 4A(n+4) \end{array}\right|,$$

which reduces to the equation

$$(-1)^{\frac{1}{2}(n-3)}\frac{(n+2)(n+1)}{8A}\,n!\,a_{n+2,\,n-2}$$
$$= a_{11}\,3.5.7\ldots(n+4)(4A)^{\frac{1}{2}(n-3)}\left\{\frac{2}{1}.\frac{1}{5.7}+\frac{4}{3}.\frac{1}{7.9}+\ldots+\frac{n-1}{n-2}.\frac{1}{(n+2)(n+4)}\right\}.$$

The series within brackets on the right-hand side of this equation is easily shown to be equal to

$$\frac{5}{36}.\frac{n+1}{n+4} - \frac{1}{12}\frac{n+1}{n(n+2)}.$$

It follows that

$$a_{n+2,\,n-2} = (-1)^{\frac{1}{2}(n-1)}(2A)^{\frac{1}{2}n}\frac{1}{(\frac{1}{2}n-\frac{1}{2})!}\left(\frac{5}{9}-\frac{2}{3}.\frac{1}{n}+\frac{1}{3}.\frac{1}{n+2}\right).$$

Putting

$$\gamma = (2A)^{\frac{1}{2}}\mu\cosh\tfrac{1}{2}\phi,$$

where $\mu = \cos\frac{1}{2}(\psi - \frac{1}{2}\pi)$ as before, the series with coefficients whose suffixes differ by four is found to be equivalent to

$$2A\mu^4\left\{\frac{5}{9}.\frac{1}{\gamma}e^{-\gamma^2} - \frac{1}{6}\frac{1}{\gamma^3}e^{-\gamma^2} - \frac{2}{3}.\frac{1}{\gamma^2}\int_0^\gamma e^{-u^2}\,du + \frac{1}{6}.\frac{1}{\gamma^4}\int_0^\gamma e^{-u^2}\,du\right\}.$$

The combination may be noted in this expression of simple harmonic terms with decreasing amplitude as $\gamma$ increases, and the diffraction integrals of FRESNEL.

A comparison of the two solutions shows that the first solution is in all respects more convenient for practical application than the second.

### *The Solution when $\kappa a$ is Small.*

We shall now consider briefly the general application of the elliptic transformation when $\kappa a$ is small. We shall assume that $\psi_0$ may have any value, but for the sake of simplicity it will be assumed that $\phi_0 = \frac{1}{2}\pi$.

Plane waves are incident in the direction DO (fig. 3). The line DO is then the asymptote of a branch of the hyperbola $\psi_0$.

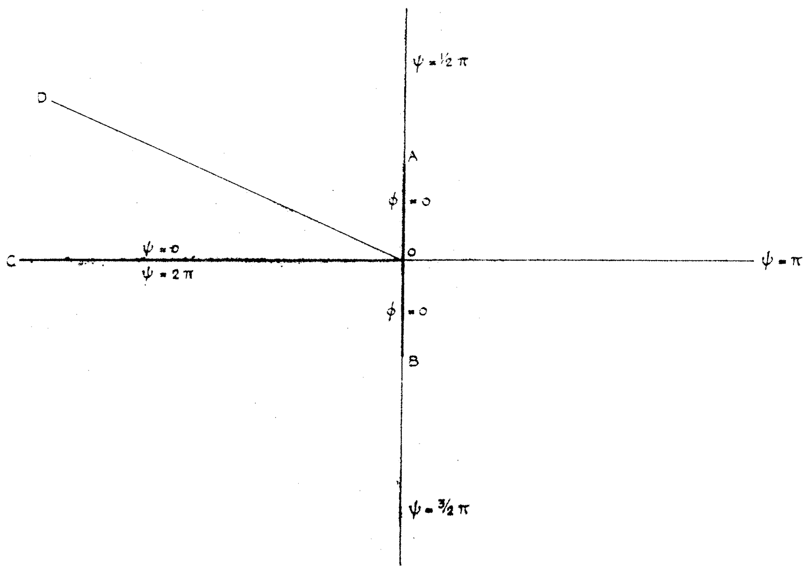


FIG. 3.

Taking F to be the solution which is being sought, the wave equation

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \kappa^2 F = 0$$

transforms into

$$\frac{\partial^2 F}{\partial \phi^2} + \frac{\partial^2 F}{\partial \psi^2} + \tfrac{1}{2}\kappa^2 a^2 (2 \sinh^2 \phi - 1 - \cos 2\psi) F = 0. \quad \ldots \quad (57)$$

Putting $F = P \,.\, Q$, where P is a function of $\phi$ only and Q a function of $\psi$ only, elementary solutions of (57) are obtained from the equations

$$\frac{\partial^2 P}{\partial \phi^2} - m^2 P + \tfrac{1}{2}\kappa^2 a^2 (2 \sinh^2 \phi - 1) P = 0, \quad \ldots \quad (58)$$

and

$$\frac{\partial^2 Q}{\partial \psi^2} + m^2 Q - \tfrac{1}{2}\kappa^2 a^2 \cos 2\psi \,.\, Q = 0. \quad \ldots \quad (59)$$

In the first place write

$$\kappa a \sinh \phi = u.$$

Equation (58) then transforms into

$$u^2 \frac{\partial^2 P}{\partial u^2} + u \frac{\partial P}{\partial u} + (u^2 - m^2) P + \kappa^2 a^2 \left(\frac{\partial^2 P}{\partial u^2} - \tfrac{1}{2} P\right) = 0. \quad . \quad . \quad . \quad . \quad . \quad (60)$$

In the second place write

$$\kappa a \cosh \phi = u.$$

Equation (58) then transforms into

$$u^2 \frac{\partial^2 P}{\partial u^2} + u \frac{\partial P}{\partial u} + (u^2 - m^2) P - \kappa^2 a^2 \left(\frac{\partial^2 P}{\partial u^2} + \tfrac{3}{2} P\right) = 0. \quad . \quad . \quad . \quad . \quad . \quad (61)$$

Solutions of the equations (59), (60) and (61) can be obtained by successive approximation, and the first approximation, which is valid when $\kappa a$ is small, is obtained from the equations

$$\frac{\partial^2 Q}{\partial \psi^2} + m^2 Q = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

and

$$u^2 \frac{\partial^2 P}{\partial u^2} + u \frac{\partial P}{\partial u} + (u^2 - m^2) P = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

Equation (63) is the well-known equation of Bessel, and, therefore, (62) and (63) are identically the same as the equations from which the elementary solutions of the wave equation, expressed in cylindrical co-ordinates, are obtained.

When $\kappa a$ is small we can, accordingly, write down the appropriate solutions at once. It is unnecessary to reproduce them here.

## Section 4.—The Parabolic Transformation.

The simplest application of equation (4) is to the problem of the incidence of plane electromagnetic waves of the prescribed type upon a parabolic cylinder.

The required transformation is

$$x + iy = \tfrac{1}{2} a (\phi + i\psi)^2,$$

in which $a$ is a constant.

In this case (4) is replaced by

$$\frac{\partial^2 F}{\partial \phi^2} + \frac{\partial^2 F}{\partial \phi^2} + 2i\kappa a \sin \phi_0 \cos \psi_0 \left(\phi \frac{\partial F}{\partial \phi} - \psi \frac{\partial F}{\partial \psi}\right) + 2i\kappa a \sin \phi_0 \sin \psi_0 \left(\psi \frac{\partial F}{\partial \phi} + \phi \frac{\partial F}{\partial \psi}\right) = 0.$$

If the normal to the incident wave front be perpendicular to the axis of $y$, $\psi_0 = 0$, and this equation reduces to

$$\frac{\partial^2 F}{\partial \phi^2} + \frac{\partial^2 F}{\partial \psi^2} + A\phi \frac{\partial F}{\partial \phi} - A\psi \frac{\partial F}{\partial \psi} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

in which $A = 2i\kappa a \sin \phi_0$.

Equation (64) is one in which the variables can be separated, and elementary solutions can be obtained by the usual method.

The function F must be zero over the surface of the parabolic cylinder which we shall denote by $\phi = \phi_1$.

The appropriate function is a solution of the differential equation

$$\frac{\partial^2 F}{\partial \phi^2} + A\phi \frac{\partial F}{\partial \phi} = 0,$$

or

$$\frac{\partial}{\partial \phi}\left(e^{\frac{1}{2}A\phi^2} \frac{\partial F}{\partial \phi}\right) = 0.$$

The solution is, therefore,

$$F = B_0 \int e^{-\frac{1}{2}A\phi^2} d\phi,$$

where $B_0$ is an arbitrary constant.

Let now

$$1/B_0 = \int_{\phi_1}^{\infty} e^{-\frac{1}{2}Au^2} du.$$

Then the complete solution is

$$u_0 = e^{i\kappa (z \cos \phi_0 + x \sin \phi_0 + ct)} B_0 \int_{\phi_1}^{\phi} e^{-i\kappa a \sin \phi_0 . u^2} du, \quad \ldots \ldots \quad (65)$$

corresponding to a wave incident at an angle $\phi_0$ with the axis of $z$.

The solution (65) is easily split up into two parts consisting of the incident wave

$$\exp i\kappa (z \cos \phi_0 + x \sin \phi_0 + ct)$$

and the scattered wave.

The foregoing represents probably the simplest possible exact solution of a problem in diffraction.

## Summary. (*Added January* 3, 1930.)


The analysis developed in the preceding paper has, as its main object, the consideration of the effect of certain obstacles upon a train of waves which is incident upon them. The obstacles are bounded by two-dimensional surfaces, and the incident waves are either plane waves of sound or plane electromagnetic waves travelling in any prescribed direction.

In the case of waves of the latter type falling upon a two-dimensional obstacle, the problem can always be solved if the electric force in the undisturbed wave front is parallel to the generating lines of the obstacle. If the obstacle is perfectly conducting the solution depends upon the discovery of a function which vanishes at every point of the surface of the obstacle. The polarisation in the diffracted wave can then be determined at any point by an application of Maxwell's electromagnetic equations.




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If the incident waves are waves of sound the function required is such that its normal gradient vanishes at every point of the surface of the obstacle.

The problem which is solved first is that of the diffraction of planes waves by two parallel semi-infinite planes. This is made possible by an application of HELMHOLTZ'S well-known transformation

$$z' = \frac{a}{\pi}(-\chi + e^{\chi}) + \omega,$$

where

$$z = x + iy,$$

$$\chi = \phi + i\psi,$$

and

$$\omega = \alpha + i\beta.$$

The distance between the two planes is $2a$; the plane $\psi = 0$ is the upper plane, and the plane $\psi = 2\pi$ is the lower. The constant $\omega$ is determined to suit our requirements. The first application of the solution is made to a consideration of the effect of the thickness of a straight edge upon FRESNEL'S diffraction phenomena in the neighbourhood of the geometrical shadow. For this purpose we assume perpendicular incidence, in which case (see equation (27) ) $\phi_0 = \psi_0 = \frac{1}{2}\pi$. In the neighbourhood of the geometrical shadow we may put $\psi = \frac{3}{2}\pi - \theta$, where $\theta$ is small.

From (27) we obtain the approximation in that neighbourhood in the form

$$\frac{u_0}{\mathrm{B}} = e^{-\frac{1}{2}\mathrm{A}e^{\phi}}\left\{1 + \theta \,.\, \frac{i\kappa a}{\pi}\right\}\int_{-\infty}^{\alpha} e^{-v^2}\,dv - \mathrm{A}^{\frac{1}{2}} e^{-\frac{1}{2}\phi}\, e^{-\frac{1}{2}\mathrm{A}e^{\phi}}\left\{\alpha^2\,(\mathrm{M} + i\mathrm{N}) + \tfrac{1}{2}\pi^{\frac{1}{2}}\int_0^{\alpha} e^{v^2}\,dv\right\}.$$

In the neighbourhood in question $\phi$ is very large, so that the second term on the right hand side of the foregoing equation is quite negligible.

On substituting $(i\frac{1}{2}\pi)^{\frac{1}{2}}\, w$ for $v$, so as to bring the integral into FRESNEL'S form, and on determining B so as to make the amplitude of the undisturbed incident wave unity, the required approximation is given by

$$u_0 = \frac{1}{\sqrt{2}}\, e^{i\frac{1}{4}\pi - i\kappa a e^{\phi}}\left\{1 + \theta \,.\, \frac{i\kappa a}{\pi}\right\}\int_{-\infty}^{\beta} e^{-i\frac{1}{2}\pi w^2}\,dw,$$

where

$$\beta = (\kappa a)^{\frac{1}{2}}\, e^{\frac{1}{2}\phi}\, \theta/\pi.$$

For the relative intensities we deduce the expression

$$\mathrm{I} = \tfrac{1}{2}\left[1 + \left(\theta \,.\, \frac{\kappa a}{\pi}\right)^2\right]\left[\left(\int_{-\infty}^{\beta}\cos \tfrac{1}{2}\pi w^2 \,.\, dw\right)^2 + \left(\int_{-\infty}^{\beta}\sin \tfrac{1}{2}\pi w^2 \,.\, dw\right)^2\right].$$

For the purposes of illustration we have taken the wave-length of the incident wave to be $1/\pi$ times the thickness $2a$ of the diffracting edge DCE (fig. 4). The arrow denotes the direction of incidence; POQ is the line along which the intensities are measured, O being the edge of the geometrical shadow drawn from C the centre of the diffracting

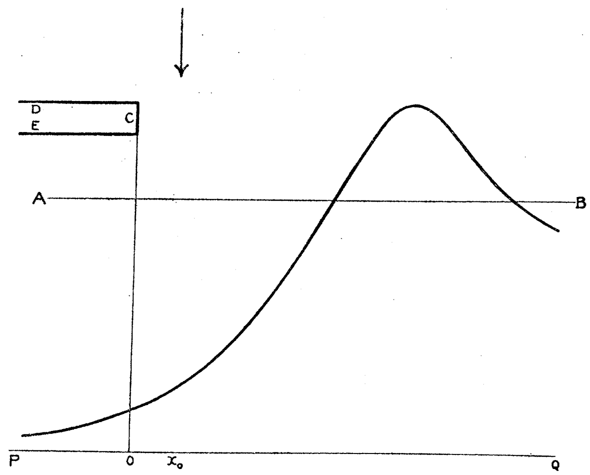


FIG. 4.

edge; and $x_0$ is the point at which the intensity is one-quarter of that in the incident wave, being situated upon the geometrical shadow when the diffracting edge is infinitely thin.

$$Ox_0 = 1 \cdot 34 \times 2a,$$

and

$$OC = 255 \times 2a.$$

AB represents the intensity in the incident wave, and the diagram is drawn to scale except in so far as the vertical distance OC is concerned.

The second application is concerned with the very interesting effect near the mouth of the two planes. Plane waves of sound are assumed to be advancing in the direction of the arrow (fig. 5) parallel to the upper side of the upper plane. The required solution is given by

$$u_0 = \tfrac{1}{2} e^{i\kappa x} + \frac{1}{\pi^{\frac{1}{2}}} e^{i\kappa x} F_0,$$

corresponding to unit amplitude in the undisturbed wave. The equation to be satisfied among the coefficients which occur in the expansion of $F_0$ is obtained by putting $\psi_0 = 0$ in equation (11). For the present purpose we have determined the values of the following coefficients :—

| | | | |
|---|---|---|---|
| $a_{11}$ | $a_{33}$ | $a_{55}$ | $a_{77}$ |
| $a_{31}$ | $a_{53}$ | $a_{75}$ | |
| $a_{51}$ | $a_{73}$ | | |
| $a_{71}$. | | | |



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The corresponding value of $F_0$ has been determined for a point P at which $\phi = 0$ and $\cos \frac{1}{2}\psi = -1/\sqrt{2}$. This value turns out to be given by

$$F_0 = -B^{\frac{1}{2}}\left[1 - \frac{79}{105}B + \frac{232}{630}B^2 - \frac{3}{35}B^3\right],$$

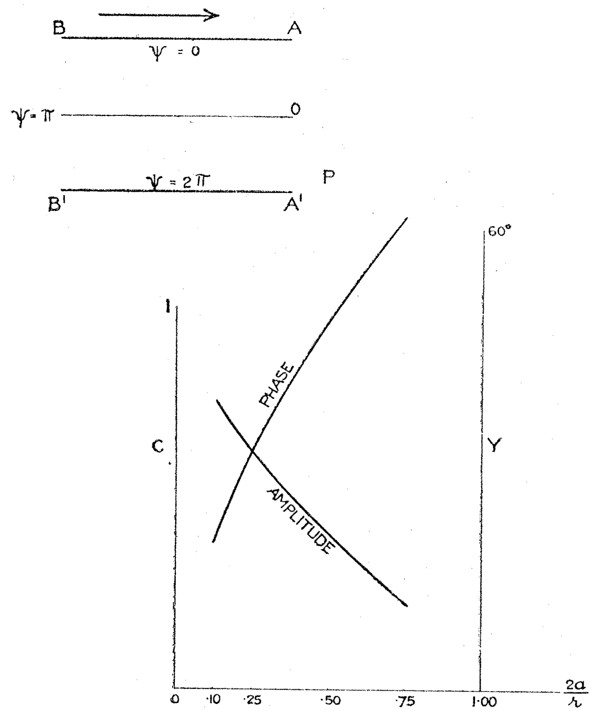


FIG. 5.

where $-iB = \beta = 2a/\lambda$ ($2a$ being the distance between the two planes and $\lambda$ the wave-length).

Realising the expression for $u_0$ we obtain finally, denoting the real part of $u_0$ by $u_{0r}$,

$$2u_{0r} = \cos \kappa x \left[1 - \frac{2\beta^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}}(P - Q)\right] + \sin \kappa x \frac{2\beta^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}}(P + Q),$$

where

$$P - Q = 1 + 0\cdot75\beta - 0\cdot37\beta^2 - 0\cdot09\beta^3$$

and

$$P + Q = 1 - 0\cdot75\beta - 0\cdot37\beta^2 + 0\cdot09\beta^3.$$

Putting $2u_{0r} = C\cos(\kappa x - \gamma)$, the diagram (fig. 5) is obtained. This diagram shows the variation of amplitude and phase at the point P, whose co-ordinates with respect to O are $x, = -0\cdot32a$, $y, = -0\cdot82a$, for a range of values of $2a/\lambda$. The amplitude and phase $\gamma$ in the undisturbed wave are, upon the scale of the diagram, two and zero respectively.

The approximation deduced for $F_0$ in the present application is valid with very little error for values of $2a/\lambda$ as great as $0\cdot75$, and it becomes of greater validity at all points near the mouth for which $y < |0\cdot82a|$.

When $\phi$ is large and negative some important approximations can be obtained which apply to points lying between the two planes. These will not be considered in detail, but we shall return briefly to the straight edge regarded as a source of diffraction phenomena, and shall consider the propagation of energy into the geometrical shadow space.

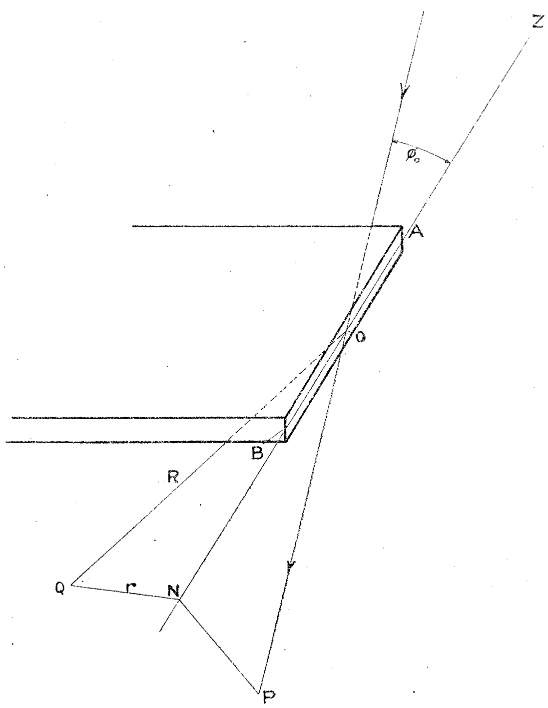

Fig. 6.

Electromagnetic waves, whose polarisation is parallel to the axis of $z$, are incident in a direction making an angle $\phi_0$ with the axis of $z$ (fig. 6). Consider any point O on the edge of the semi-infinite plane AB, and let OP be the direction of the incident waves. Rotate OP upon the surface of a zone of semi-angle $\phi_0$ till OP comes into the position OQ, which lies well within the geometrical shadow. Let QN be perpendicular to OZ. Then it is shown that at Q the electric force in the scattered wave is sensibly perpendicular to OQ and lies in the plane OQN, while the magnetic force at Q is perpendicular to the plane OQN. Energy is therefore propagated within the shadow as though from a source at O.

In fig. 3 another interesting problem is presented by the diffraction of plane waves impinging upon a semi-infinite plane with a plane lamina at the end. In this case we require the elliptic transformation given by

$$x + iy = a \sinh (\phi + i\psi),$$

in which $\phi, = 0$, represents the lamina and $\psi, = 0$ and $2\pi$ respectively, represents the two sides of the semi-infinite plane. The incident waves are sound waves and the angle of incidence is such that $\psi_0 = \frac{1}{2}\pi$.

The first point to notice in the solution is the extremely small effect of the lamina upon Fresnel's diffraction phenomena. An illustration of the motion near the lamina is obtained by the determination of the values of the coefficients in the expansion of $F_0$ by means of the equation (49).

Putting $\phi_0$, the angle which the normal to the incident wave front makes with the axis of $z$, equal to $\frac{1}{2}\pi$, that is, assuming perpendicular incidence upon the semi-infinite plane, the complete solution is given by.

$$u_0 = \tfrac{1}{2}e^{i\kappa y} + \tfrac{1}{2}e^{-i\kappa y} + \frac{2}{\pi^{\frac{1}{2}}}\{e^{i\kappa y}F_i + e^{-i\kappa y}F_r\},$$

where

$$F_i = a_{11}\cosh\tfrac{1}{2}\phi\cos\tfrac{1}{2}(\psi - \tfrac{1}{2}\pi) + \ldots$$

and

$$F_r = a_{11}\cosh\tfrac{1}{2}\phi\cos\tfrac{1}{2}(\psi + \tfrac{1}{2}\pi) + \ldots,$$

Now the approximation contemplated requires the evaluation of a number of the " $a$'s," each " $a$ " being a function of $\frac{1}{2}A, = i\kappa a$. These are as follows :—

| | | | | |
|---|---|---|---|---|
| $a_{11}$ | $a_{33}$ | $a_{55}$ | $a_{77}$ | $a_{99}$ |
| $a_{51}$ | $a_{73}$ | $a_{95}$ | $a_{11,7}$ | |
| $a_{71}$ | $a_{93}$ | $a_{11,5}$ | | |
| $a_{91}$ | $a_{11,3}$ | | | |
| $a_{11,1}$. | | | | |

Let us now consider a point P close to the lamina where $\phi$ is sensibly zero and $\psi = \frac{3}{2}\pi + \varepsilon$, $\frac{1}{2}\varepsilon$ being moderately small. We obtain, for the real part $u_{0r}$ of $u_0$,

$$u_{0r} = \cos\kappa y - \frac{2\beta^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}}[\cos\kappa y\,(P - R) + \sin\kappa y\,(Q + S)],$$

where $\beta = \kappa a\cos^2\frac{1}{2}\varepsilon$, and P, Q, R, S are functions of $\kappa a$ and $\varepsilon$.

When $\frac{1}{2}\varepsilon = 16°$, which we take for illustration,

$$P = 1\cdot270 - 0\cdot372\beta^2 + 0\cdot170\beta^4$$

$$R = S = -0\cdot333\beta + 0\cdot214\beta^3 - 0\cdot081\beta^5$$

$$Q = 0\cdot730 - 0\cdot372\beta^2 + 0\cdot170\beta^4.$$

Let

$$u_{0r} = C\cos(\kappa y + \gamma);$$

then the diagram (fig. 7) shows the values of C and $\gamma$, for a range of values of $2a/\lambda$, at the point P, where $CP = 0\cdot85a$.

The amplitude GH of the incident wave is unity and the phase $\gamma$ is zero.

As a final illustration of perpendicular incidence we take the following approximation to the solution, viz. :—

$$u_{0r} = \cos\kappa y + \frac{2}{\pi^{\frac{1}{2}}}(\kappa a)^{\frac{1}{2}}\cosh\tfrac{1}{2}\phi\,(\cos\kappa y\cos\tfrac{1}{2}\psi - \sin\kappa y\sin\tfrac{1}{2}\psi).$$

This expression is a good approximation when $\kappa a$ is small and when

$$\frac{2}{\pi^{\frac{1}{2}}}(\kappa a)^{\frac{1}{2}} \cosh \tfrac{1}{2}\phi$$

is not much greater than $0 \cdot 5$.

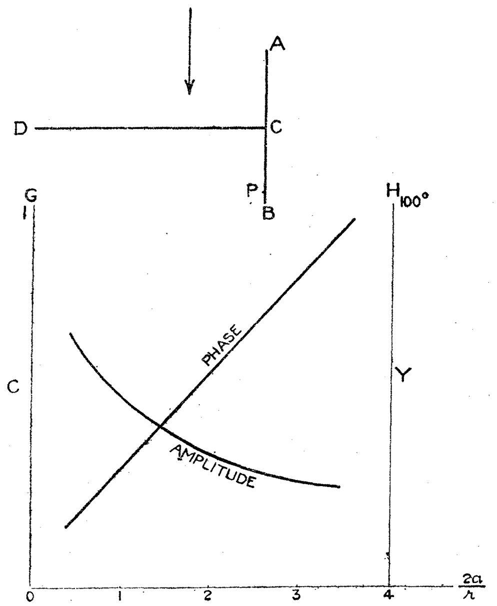


FIG. 7.

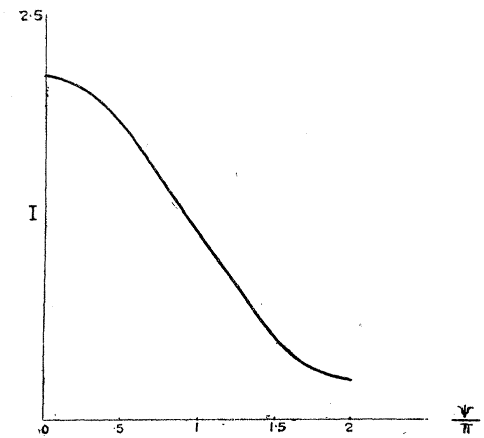


FIG. 8.

Put this function equal to $0 \cdot 5$ and determine the intensity I for all values of $\psi$. Then

$$I = 1 \cdot 25 + \cos \tfrac{1}{2}\psi.$$

Fig. (8) shows the intensity distribution around the lamina, the intensity in the incident wave being unity.



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